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OF NOTED MATHEMATICIANS, ETC.

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ON SYMMETRIC FUNCTIONS.

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[Presented at the October meeting of the American Mathematical Society.]

The object of the following paper is to call attention to certain methods of treating symmetric functions of the roots of an equation from two quite different standpoints. From the first standpoint the symmetric function as a whole as expressed in terms of the coefficients of the equation is considered; from the second, the isolation of one of these terms with its numerical coefficient is the object of investigation. Accordingly the paper is divided into two chief divisions. We proceed to the first.

I. SYMMETRIC FUNCTIONS AS A WHOLE.

It is not the writer's purpose to reproduce the many formulas and methods which are already given in works on algebra for expressing symmetric functions as a whole. Only one of them will be noticed, and attention called to two others, as follows:

A. FORMULAS.

1. BRIOSCHI'S FORMULA IN TERMS OF THE s'S.

(1). *Statement of the Formula.*

In 1854 in the *Annali di Tortolini* (t. 5, pp. 427-8), Brioschi gave without proof the following formula:

$$\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=\begin{vmatrix} u_{11} & u_{12}\ldots\ldots u_{1n} \\ u_{21} & u_{22}\ldots\ldots u_{2n} \\ \ldots & \ldots\ldots\ldots\ldots \\ u_{n1} & u_{n2}\ \ldots\ldots\ u_{nn} \end{vmatrix}$$

where $u_{11}=s_{p_1}$, $u_{22}=s_{p_2}$, $u_{12}u_{21}=s_{p_1+p_2}$, $\ldots\ldots u_{rs}u_{st}u_{tu}\ldots\ldots u_{xr}=s_{p_r+p_s+p_t\ldots,p_r}$.

(2). *An Inadequate Statement.*

This formula seems to have lost the clearness of Brioschi's statement in Faà di Bruno's *Binäre Forman*, p. 8, so that the statement there made is inadequate. The writer has given a correction and proof of the formula (June-July number, 1898, pp. 161-4, of the MONTHLY).

(3). *Critical Value of the Formula.*

Taken in connection with the formula

$$s_r=\frac{(-1)^r}{a_{0^r}}\begin{vmatrix} a_1 & a_0 & 0 & 0\ldots\ldots\ldots\ldots0 \\ 2a_2 & a_1 & a_0 & 0\ldots\ldots\ldots\ldots0 \\ 3a_3 & a_2 & a_1 & a_0\ \ 0\ldots\ldots\ldots0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots a_0 \\ ra_r & a_{r-1} & a_{r-2} & \ldots\ldots\ldots\ldots a_1 \end{vmatrix}$$

of degree r and weight r as seen by developing in terms of the elements of the last line, it affords an example of a complete theoretical solution of the problem of expressing the symmetric function $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$ in terms of the coefficients of the equation, and gives at once the theorems concerning rationality and weight; but on account of the theorem concerning order it is clear that this expression must contain many superfluous terms which destroy in the working out, and is therefore little adapted to practical purposes.

2. GORDAN'S FORMULA IN TERMS OF THE a'S.

(1). *Statement of the Formula.*

The following theorem is due to Professor Gordan, and was stated to the writer by him. If we multiply the alternating function

$$D=\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2\ldots\ldots\alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2\ldots\ldots\alpha_2^{n-1} \\ \ldots & \ldots & \ldots\ldots\ldots \\ 1 & \alpha_n & \alpha_n^2\ldots\ldots\alpha_n^{n-1} \end{vmatrix}$$

by $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$ supposing the p's to be all different, [If k_1 of the p's are p_1, k_2, p_2, $\ldots\ldots k_r$, p_r, we must divide the result by $k_1!\,k_2!\ldots\ldots k_r!$ to obtain Σ.] the result is the same as if we apply the n exponents of Σ to the columns of D in all n ! permutations of the same, and take the sum of the n ! determinants so resulting. This operation Professor Gordan indicates as $[p_1p_2\ldots\ldots p_n]$ and its

expansion as $\Sigma(p_{i_1}\, p_{i_2}+1 \ldots \ldots p_{i_n}+n-1)$, where $i_1, i_2 \ldots\ldots i_n$ form a permutation of the numbers 1, 2, ...n. Thus we have the formula

$$D\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}=[p_1 p_2 \ldots\ldots p_n]=\Sigma(p_{i_1} p_{i_2}+1 \ldots\ldots p_{i_n}+n-1).$$

(2). *Application of the Formula.*

To apply it we notice that the matrices

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1{}^2 \ldots\ldots \alpha_1{}^{2n-1} \\ 1 & \alpha_2 & \alpha_2{}^2 \ldots\ldots \alpha_2{}^{2n-1} \\ \ldots & \ldots & \ldots \\ 1 & \alpha_n & \alpha_n{}^2 \ldots\ldots \alpha_n{}^{2n-1} \end{vmatrix} \text{ and } \begin{vmatrix} a_n & a_{n-1} & a_{n-2} \ldots\ldots a_0 & 0 & 0 \ldots\ldots 0 \\ 0 & a_n & a_{n-1} \ldots\ldots a_1 & a_0 & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 \ldots 0 & a_n \ldots\ldots & \ldots & \ldots a_1 a_0 \end{vmatrix}$$

correspond, and from the theorem that "the corresponding determinants of corresponding matrices are proportional,"* that

$$D=\rho \begin{vmatrix} a_0 & 0 & 0 \ldots\ldots 0 \\ a_1 & a_0 & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots \\ a_{n-1} & \ldots\ldots & a_1 a_0 \end{vmatrix} =\rho a_0{}^n.$$

When, as often happens, any of the numbers p_r+i_r, p_s+i_s, in a parenthesis, are equal, such symbolic parenthesis reduces to zero, because the determinant thereby signified has at least two columns equal. The values of the non-vanishing symbols are then read off from the second matrix by the help of the theorem concerning corresponding matrices, the factor ρ being added and the correct sign factor of the corresponding determinant. The factor ρ will then divide out and the result gives

$$a_0{}^n\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}$$

which equals the algebraic sum of a number of determinants in the a's of the nth order.

(3). *Proof of the Formula.*

I have proved this theorem as follows: We will begin by considering some simple examples.

a. It is required to find $\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}$ for a quadratic equation $a_0x^2+a_1x+a_2=0$. The matrices are

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1{}^2 & \alpha_1{}^3 \\ 1 & \alpha_2 & \alpha_2{}^2 & \alpha_2{}^3 \end{vmatrix} \text{ and } \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \end{vmatrix}.$$

$$D=\begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix}=\rho\begin{vmatrix} a_0 & 0 \\ a_1 & a_0 \end{vmatrix}=\rho a_0{}^2.$$

---

*Gordan, Determinants, p. 95.

We indicate $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}=(p_1p_2)$. We perform the operation indicated on the left hand side in detail. We have

$$(\alpha_2-\alpha_1)(\alpha_1^{p_1}\alpha_2^{p_2}+\alpha_1^{p_2}\alpha_2^{p_1})=\alpha_1^{p_1}\alpha_2^{p_2+1}+\alpha_1^{p_2}\alpha_2^{p_1+1}-\alpha_1^{p_1+1}\alpha_2^{p_2}-\alpha_1^{p_2+1}\alpha_2^{p_1}$$

$$=\begin{vmatrix}\alpha_1^{p_1} & \alpha_1^{p_2+1}\\ \alpha_2^{p_1} & \alpha_2^{p_2+1}\end{vmatrix}+\begin{vmatrix}\alpha_1^{p_2} & \alpha_1^{p_1+1}\\ \alpha_2^{p_2} & \alpha_2^{p_1+1}\end{vmatrix}$$

by taking the first and fourth, and the second and third terms together. If we had multiplied

$$\begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix} \text{ by } \Sigma\alpha_1^{p_1}\alpha_2^{p_2} \text{ as } \begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix}\alpha_1^{p_1}\alpha_2^{p_2}+\begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix}\alpha_1^{p_2}\alpha_2^{p_1},$$

we should have obtained

$$\begin{vmatrix}\alpha_1^{p_1} & \alpha_1^{p_1+1}\\ \alpha_2^{p_2} & \alpha_2^{p_2+1}\end{vmatrix}+\begin{vmatrix}\alpha_1^{p_2} & \alpha_1^{p_2+1}\\ \alpha_2^{p_1} & \alpha_2^{p_1+1}\end{vmatrix}.$$

If we compare with the previous result, we see that we could convert one into the other by exchanging in each the two secondary diagonal terms which is permitted. We choose the result first obtained and symbolize it by $(p_1p_2+1)+(p_2p_1+1)$, where in (p_1p_2+1) p_1 is the exponent of the first, and p_2+1 that of the second column of the determinant for which it stands. We write completely

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}=(p_1p_2)=(p_1p_2+1)+(p_2p_1+1).$$

b. We will apply this formula to the calculation of the symmetric functions required by the resultant of two quadratic forms.

$$fx=a_0x^2+a_1x+a_2=a_0(x-\alpha_1)(x-\alpha_2).$$
$$\phi x=b_0x^2+b_1x+b_2=b_0(x-\beta_1)(x-\beta_2).$$

The resultant of these forms is

$$a_0^2(b_0\alpha_1^2+b_1\alpha_1+b_2)(b_0\alpha_2^2+b_1\alpha_2+b_2)=$$
$$a_0^2(b_0^2\Sigma\alpha_1^2\alpha_2^2+b_0b_1\Sigma\alpha_1^2\alpha_2+b_0b_2\Sigma\alpha_1^2+b_1^2\Sigma\alpha_1\alpha_2+b_1b_2\Sigma\alpha_1+b_2^2).$$

c. We comprehend the results together in the following table. The sign of the determinant which corresponds to $(\alpha_1^\lambda\beta_1^\mu)$ is by the theorem of corresponding matrices,

$$(-1)^{(1+2)+(\lambda+1+\mu+1)}=(-1)^{\lambda+\mu+1}.$$

We also observe that $(\lambda\lambda)=0$, and do not write 0.

Function	$[p_1p_2]$	$\begin{matrix}(p_1 & p_2+1) \\ (p_2 & p_1+1)\end{matrix}$	Determinants	Sign	Result
$2a_0^2\Sigma\alpha_1^2\alpha_2^2$	[2 2]	2(2 3)	$2\begin{vmatrix} a_2 & a_1 \\ 0 & a_2 \end{vmatrix}$	$(-1)^6=1$	$2a_2^2$
$a_0^2\Sigma\alpha_1^2\alpha_2$	[2 1]	(2 2) (1 3)	$\begin{vmatrix} a_2 & a_0 \\ 0 & a_1 \end{vmatrix}$	$(-1)^5=-1$	$-a_1a_2$
$a_0^2\Sigma\alpha_1^2$	[2 0]	(2 1) (0 3)	$-\begin{vmatrix} a_2 & 0 \\ 0 & a_0 \end{vmatrix}$ $\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}$	$-(-1)^4=-1$ $(-1)^4=1$	$-2a_0a_2+a_1^2$
$2a_0^2\Sigma\alpha_1\alpha_2$	[1 1]	2(1 2)	$2\begin{vmatrix} a_2 & 0 \\ 0 & a_0 \end{vmatrix}$	$(-1)^4=1$	$2a_2a_0$
$a_0^2\Sigma\alpha_1$	[1 0]	(1 1) (0 2)	$\begin{vmatrix} a_1 & 0 \\ a_2 & a_0 \end{vmatrix}$	$(-1)^3=-1$	$-a_0a_1$

d. Substituting these values we have for the final result the function of the coefficients of the two quadratics

$$b_0^2a_2^2-b_0b_1a_1a_2+b_0b_2(a_1^2-2a_0a_2)+b_1^2a_0a_2-b_1b_2a_0a_1+b_2^2a_0^2.$$

e. We will now take another step and consider $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. We wish to show that we obtain the same result when we apply the exponents in all possible permutations to the columns of D that we obtain when we apply them to the rows, and obtain $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. To prove this we need only show that to every term of the second formation corresponds exactly the same term in the first formation. First, it is clear that the number of terms in each formation is $(3\,!)^2=36$. Next, take a term like $\alpha_1^{p_3+1}\alpha_2^{p_1+2}\alpha_3^{p_2}$ of the second formation. It must have come from

$$\begin{vmatrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \\ \alpha_3^0 & \alpha_3 & \alpha_3^2 \end{vmatrix} \begin{matrix} p_3 \\ p_1=d' \\ p_2 \end{matrix}$$

which signifies that D has been multiplied by $\alpha_1^{p_3}\alpha_2^{p_1}\alpha_3^{p_2}$ of $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$, the first row by $\alpha_1^{p_3}$, the second by $\alpha_2^{p_1}$, and the third by $\alpha_3^{p_2}$, and here as the exponents come to be applied to the rows as the result of the multiplication, they are legitimately applied. Again write D changing columns into rows, and obtain

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}$$

Next inquire by what substitution $\alpha_1^{p_3+1}\alpha_2^{p_1+2}\alpha_3^{p_2}$ came from the previous determinant d'. We see that it corresponds to 1st line 2d column, 2d line 3d column, 3d line 1st column, or to the substitution $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$. Now in order that we may arbitrarily affix the exponents to the rows of D as last written, which is the same as affixing them to the columns as before written, and obtain this term, at least numerically, we must have p_3 with the second line, p_1 with the third, and p_2 with the first. Thus we must have

$$\left|\begin{matrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{matrix}\,\right|\begin{matrix} p_2 \\ p_3 \\ p_1 \end{matrix} = d_1$$

which signifies that the exponents p are to be arbitrarily affixed to those of the corresponding line.

What substitution now gives the term in this determinant? We see that it corresponds to 2d line 1st column, 3d line, 2d column, 1st line 3d column, or to $\begin{pmatrix}2 & 3 & 1\\ 1 & 2 & 3\end{pmatrix}$; and this last substitution is the reciprocal of the preceding $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$ and has the same sign with it. We see farther that to every term of d' corresponds one and only one determinant like d_1, and in d_1 there is one and only one term equal to the given term of d'. To the six terms of d' correspond the six determinants of d_1, d_2, ...d_6; to the six terms of d'' correspond the same six determinants d_1, ...d_6, and so on; to the 36 terms of the six determinants d', d'', ... d^{vi}, correspond the six determints d_1, ...d_6, and conversely to the 36 terms of the six determinants d_1, ...d_6, correspond the six determinants d', d'', ...d^{vi}. Thus the 36 terms of the one set are equal to the 36 terms of the other set, and the results of the two operations are identical.

f. We may express the result of this example more briefly if we write:

$$\left|\begin{matrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \\ \alpha_3^0 & \alpha_3 & \alpha_3^2 \end{matrix}\,\right|\begin{matrix} p_{i_1} \\ p_{i_2} \\ p_{i_3} \end{matrix}$$

and take any one of the 36 terms obtained by applying the 3 ! substitutions $\begin{pmatrix}1 & 2 & 3\\ j_1 & j_2 & j_3\end{pmatrix}$ where the j's are the numbers of the columns, to the six determinants that arise from the 3 ! permutations of the i's which form a permutation of the numbers 1, 2, 3. We obtain a term of the form

$$\pm\alpha_1^{p_{i_1}+j_1-1}\alpha_2^{p_{i_2}+j_2-1}\alpha_3^{p_{i_3}+j_3-1}.$$

[To be continued.]

ENCYKLOPAEDIE DER MATHEMATISCHEN WISSENSCHAFTEN.

By DR. GEORGE BRUCE HALSTED.

Mit Unterstuetzung der Akademicen der Wissenschaften zu Muenchen und Wien und der Gesellschaft der Wissenschaften zu Goettingen, herausgegeben von *H. Burkhardt* und *W. F. Meyer*. Band 1. Heft 1. Leipzig, Teubner. 1898. Pages 1—112.

This is an undertaking of extraordinary importance and promise. Its aim is to give a conservative presentation of the assured results of the mathematical sciences in their present form, while, by careful and copious references to the literature, giving full indications regarding the historic development of mathematical methods since the beginning of the nineteenth century. The work begins with 27 pages on the foundations of arithmetic by Hermann Schubert of Hamburg. Schubert's reputation was made by his remarkable book on enumerative geometry. He has since applied modern ideas in an elementary arithmetic, and is known in America as a contributor to the *Monist*. Unfortunately, Schubert has made in public some strange slips. In an article "On the nature of mathematical knowledge," in the *Monist*, Vol. 6, page 295, he says: "Let me recall the controversy which has been waged in this century regarding the eleventh axiom of Euclid, that only one line can be drawn through a point parallel to another straight line. The discussion merely touched the question whether the axiom was capable of demonstration solely by means of the other propositions or whether it was not a special property, apprehensible only by sense-experience, of that space of three dimensions in which the organic world has been produced and which therefore is of all others alone within the reach of our powers of representation. The truth of the last supposition affects in no respect the correctness of the axiom but simply assigns to it, in an epistemological regard, a different status from what it would have if it were demonstrable, as was one time thought, without the aid of the senses, and solely by the other propositions of mathematics."

If Schubert had written this seventy-five years ago it might have been pardonable. Just at the beginning of this century Gauss was trying to prove this Euclidean parallel-postulate. Even up to 1824 he was in Schubert's state of mind, for he then writes Taurinus: "Ich habe daher wohl zuweilen *in Scherz* den Wunsch geaeussert, dass die Euclidische Geometrie nicht die Wahre waere." But the joke had even then gone out of the matter if Gauss had but known it, for in 1823 Bolyai Janos had written to his father, "from nothing I have created a wholly new world." Of the geometry of this world as given also by Lobachevski, Clifford wrote: "It is quite simple, merely Euclid without the vicious assumption." But this assumption is only vicious if supposed to be "apprehensible by sense-experience" or "demonstrable by the aid of the senses." That "the organic world has been produced" in Euclidean space can never be demonstrated

in any way whatsoever. On the other hand, the mechanics of actual bodies might be shown by merely approximate methods to be non-Euclidean.

Therefore Schubert's contribution on the foundations of arithmetic may fairly be read critically. He begins with counting, and defines number as the result of counting. This is in accord with the theory that their laws alone define mathematical operations, and the operations define the various kinds of number as their symbolic outcome. There is no word of the primitive number-idea, which is essentially prior to counting and necessary to explain the cause and aim of counting. This primitive number-idea is a creation of the human mind, for it only pertains to certain other creations of the human mind which I call artificial individuals. The world we consciously perceive is a mental phenomenon. Yet certain separable or distinct things or primitive individuals we cannot well help believing to subsist somehow 'in nature' as well as in conscious perception. Now by taking together certain of these permanently distinct things or natural individuals the human mind makes an artificial individual, a conceptual unity.

Number is primarily a quality of such an artificial individual. The operation of counting was made to apply to such an individual to identify it with one of a standard set of such artificial individuals, and so to get the exact shade of its numeric quality. These standard individuals were primarily sets of fingers. Then came the written standard set, *e. g.* III, or $1+1+1$; and finally the written symbol 3. Such symbols serve to represent and convey the numeric quality. The word number is applied indiscriminately to the quality or idea and to its symbol.

Schubert tells us that in antiquity the Romans represented the numbers from one to nine by rows of strokes, as 4 is still represented on our watches ; while the Aztecs used to put together single circles for the numbers from one to nineteen. I have seen Japanese use columns of circles in the same way. Thus also our striking clocks convey a numeric quality by a group possessing it. But the number pertaining to a group or artificial individual is far from being the simple notion it seems. If numbers are used to express exactly this definite attribute of finite systems they are called cardinal numbers.

Schubert's first sentence is : Dinge *zaehlen* heisst, sie als gleichartig ansehen, zusammen auffassen, und ihnen einzeln andere Dinge zuordnen, die man auch als gleichartig ansieht. This may be rendered: "*To count* things means, to consider them as alike, to take them together, and to associate them singly to other things which one also considers as alike." I would prefer to say : "To count distinct things means to make of them an artificial individual or group and then to identify its elements with those of a familiar group."

When the mind of man made these artificial individuals, they were found to possess a sort of property or quality which was independent of the distinctive marks of the natural individuals composing them, also independent of the order or sub-association of these natural individuals. Whether the artificial individual were made of a church, a noise, and a pain, or made of three peas, or composed of two eyes and a nose, it had one certain quality, it was a triplet.

I see no necessity for Schubert to consider the church as like the noise and the pain. Again, the individuals of the familiar group used in the count need not be alike. Even the individuals used by a clock in counting differ ordinarily, and when we follow the count of the clock we use words all different. The primitive written number is such a picture of a group of individuals as represents their individual existence and nothing more, *e. g.* III; so however different they may be, this number is independent of the order in which they are associated with its elements.

Schubert wastes three sentences on the so-called concrete number, *benannte* Zahl. Three quails is not a number, but is a particular bevy.

His §2 *Addition*, he begins thus: "If one has two groups of units such that not only all units of each group are alike, but that also each unit of the one group is like each unit of the other group, etc." All this likeness and alikeness seems unnecessary. Any two groups may be thought into one group. Any two primitive numbers may be added.

In section 5, Peacock's Principle of Permanence is given it Hankel's general form: The combination of two numbers by any defined operation is a number, such that the combination may be handled as if it gave one of the previously defined numbers. New kinds of numbers, like all numbers, are defined by the operations from which they result. Thus are introduced zero and negative numbers, and later, the fraction. After this all is easy to the end of Schubert's contribution. It only remains to point out, as of especial importance, that from beginning to end not the slightest mention is made of measurement. Not a word is wasted on people who do not clearly see that number is long prior to measurement.

The second section of the Encyklopaedie is "Kombinatorik" by E. Netto. This is a part of mathematics which never fulfilled the hopes of the school which was lost in it during the early part of this century. Of the most comprehensive monographs the last two are in 1826 and 1837. For us it has gone over into determinants, and more than half of Netto's article is devoted to determinants. This article is particularly valuable from a bibliographic and historic point of view.

The third section is "Irrationalzahlen und Konvergenz unenlicher Prozesse," by A. Pringsheim. It begins on page 47, and goes past the end of the Heft. This is a modern subject, of intense living interest. How entirely modern it is might not be suspected by readers of such sentences in Cajori's excellent history of mathematics as those on page 70; "The first incommensurable ratio known seems to have been that of the side of a square to its diagonal, as $1:\sqrt{2}$. Theodorus of Cyrene added to this the fact that the sides of squares represented in length by $\sqrt{3}$, $\sqrt{5}$, etc., up to $\sqrt{17}$, and Theaetetus, that the sides of any square, represented by a surd, are incommensurable with the linear unit."

Now in fact Theodorus and Theaetetus made no representation whatever of the length of these sides, simply saying, *e. g.* that the side of the square whose area is 3 is incommensurable with the side of the square whose area is one. For

Euclid there was no such ratio as $1:\sqrt{2}$; for 1 is a number and so if it could have had a ratio to $\sqrt{2}$ this would have been a number. But Euclid, Book X, Proposition 7 is: "Incommensurable magnitudes are not to one another in the ratio of one number to another number." The Hindus were the first to recognize the existence of irrational numbers. Even through the middle ages and the renaissance they were absurd fictions, "numeri surdi," a designation attributed to Leonardo of Pisa. The first writer to genuinely treat them was Stifel (1544), and even he had not completely freed himself from the older terminology, since he says: "sic irrationalis numerus non est verus atque lateat sub quadam infinitatis nebula." In reference to the next step, the conceiving of ratio as number, Pringsheim says, page 51, "Hatte schon *Descartes* beliebige *Streckenverhaeltnisse* mit *einfachen Buchstaben* bezeichnet, und damit *wie mit Zahlen* gerechnet, etc." Bnt here I think the careful German has slipped.

In regard to just this point a common error is still widespread, which we see in the following, read before Sections A and B of the American Association for the Advancement of Science, 1891:

"The doctrine of Descartes was, that the algebraic symbol did not represent a concrete magnitude, but a mere number or ratio, expressing the relation of the magnitude to some unit. Hence that the product of two quantities is the product of ratios, . . . ; that the powers of a quantity are ratios like the quantity itself," etc.

That every statement here quoted is a mistake will be instantly seen from the following, taken from pages numbered 297—9 of the *original edition* of Descartes' Geometrie, 1637, a copy of which (perhaps unique on this continent) I have had the good fortune to possess since my student days (1876).

"Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, que sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision: Ainsi n' at' on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres, ou en oster; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer que quatriesme. qui soit a l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité est a l'autre, ce qui est le mesme que la Diuision; au enfin trouuer vne, ou deux, ou plusieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui est le mesme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligible. * * *

"Mais souuent on n'a pas besioin de tracer ainsi ces lignes sur le papier, & il suffist de les designer par quelques lettres, chascune par vne seule. Comme pour adiouster la ligne BD a GH, ie nomme l'vne a & l'autre b, & escris $a+b$; Et $a—b$, pour soustraire b d' a; Et ab, pour les multiplier l'vne par l' autre; Et $\frac{a}{b}$, pour diuiser a par b; Et aa, ou a^2, pour multiplier a par soy mesme; Et a^3, pour le multiplier encore vne fois par a, & ainsi a l' infini; Et $\sqrt{a^2+b^2}$, pour tirer la racine quarrée d' a^2+b^2; Et $\sqrt{C.a^3-b^3+abb}$, pour tirer la racine cubique d' a^3-b^3+abb, & ainsi des autres.

"Ou il est a remarquer que par a^2 ou b^3 ou semblables, ie ne concoy ordinairement que des lignes toutes simples, encore que pour me seruir des noms vsités en l'Algebre, ie les nomme des quarrés, ou des cubes, &c."

Thus what Descartes really did was to make a geometric algebra, in which, however, the product of two *sects* (Strecken) was not a rectangle but a sect; the product of three sects not a cuboid but a sect. Here Descartes represents by the single letters a, b, sects, Strecken, not *Streckenverhaeltnisse.* Descartes does not here pass beyond Euclid's representation of the ratio of two magnitudes by two other magnitudes, does not reach the conception of the systematic representation of the ratio of two magnitudes by one magnitude, that one magnitude to be always interpreted as a number. This radical innovation, the creation of this epoch-marking paradox, is due to Newton. Newton takes this vast step explicitly and consciously. The lectures which he delivered as Lucasian professor at Cambridge were published under the title "Arithmetica Universalis." At the beginning of his Arithmetica Universalis he says, page 2, "Per Numerum non tam multitudinem unitatum quam abstractam quantitatis cujusvis ad aliam ejusdem generis quantitatem quae pro unitate habetur rationem intelligimus." [In quoting this, Pringsheim, page 51, misses the first word. He omits the *Per.*] As Wolf puts it, (1710) "Number is that which is to unity as a piece of a straight line [a sect] is to a certain other sect." Thus the length of any sect is a real number, and the length of any possible sect incommensurable with the unit sect is an irrational number. Says Hayward in his Vector Algebra (1892), page 5, "Number is essentially *discrete* or *discontinuous*, proceeding from one value to the next by a finite increment or jump, and so cannot, except in the way of a limit, represent, relatively to a given unit, a continuous magnitude for which the passage from one value to another may always be conceived as a *growth* through every intermediate value." But the moment we accept Newton's definition of number it takes on whatever continuity is possessed by the sect. However, from this alone does not follow that for every irrational there is a sect whose length would give that irrational. G. Cantor was the first to bring out sharply that this is neither self-evident nor demonstrable, but involves an essential pure geometric assumption. To free the foundations of general arithmetic from such *geometric* assumption, G. Cantor and Dedekind each developed his pure arithmetic theory of the irrational. Professor Fine in his "Number-System of Algebra" seems to miss this point completely. He gives, page 42, what purports to be a demonstration that "Corresponding to every real number is a point on the line, the distance of which from the null-point is represented by the number," without any mention of the geometric assumption necessary, and then proceeds, page 43, to borrow the continuity of his number system from the näively supposed continuity of the line, the very thing for the avoidance of which G. Cantor and Dedekind made their systems. Says Dedekind, "Um so schoener erscheint es mir, dass der Mensch ohne jede Vorstellung von messbaren Groessen, und zwar durch ein endliches System einfacher Denkschritte sich sur Schoepfung des reinen, stetigen Zahlenreiches aufschwingen kann; und erst mit diesem Huelfsmittel wird es ihm nach meiner Ansicht moeglich, die Vorstellung von stetigen Raume zu einer deutlichen auszubilden."

Austin, Texas.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

103. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Find the proceeds of a note discounted at a bank for 10 years at 10%. What is the meaning of the result ?

I. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

The proceeds would be nothing. The meaning is that the method is erroneous and unjust. It was invented to evade the usury laws. I have thought that the court which first sustained the method could not have been well versed in mathematical principles.

II. Solution by the PROPOSER.

The bank discount upon any sum exceeds the true discount for the same time and rate, by the interest upon the true discount for the given time. Bank discount then is of the nature of compound interest, for a banker in discounting a note not only deducts the interest which is to accrue upon it from that time until the date of maturity, but also charges a certain per cent. for his services, which per cent. is the interest upon the interest which is to accrue. Now the true discount of any sum for 10 years at 10 per cent. is just half that sum, and as the amount will double itself at simple interest for 10 years at 10 per cent., the discount which the banker deducts for that time, plus the interest which he charges, is exactly equal to the face of the note, neglecting the three days of grace. The note in this light is worthless in spite of the fact that its present worth is half of its face. The case is of course a very improbable one but it is simply an illustration of the rapidity with which compound interest will accumulate, although the interest here is not compound in strict sense but accumulates slower.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; ELMER SCHUYLER, High Bridge, N. J.; CHAS. C. CROSS, Libertytown, Md., and ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Ia.

Days of grace are no longer in use. Since bank discount is simple interest charged when loan is made the proceeds would be $0.00.

This means that the bank would get twice the amount loaned, while the borrower would have the use of no money. Such a transaction is hardly possible between business men.

Solutions of problem 102 were received too late for credit in the December number from *JOSIAH H. DRUMMOND, P. S. BERG, ALOIS F. KOVARIK, P. H. PHILBRICK, ELMER SGHUYLER, GUY B. COLLIER, G. BRECKENRIDGE,* and *WALTER H. DRANE.*

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithica, N. Y.

Solve by quadratics, $x^2+y=7......(1)$,
$x+y^2=11......(2)$.

IX. Solution by S. F. NORRIS, Professor of Mathematics and Astronomy, Baltimore City College, Baltimore, Md.

$x^2+y=7......(1)$, $x+y^2=11......(2)$.

Eliminating y, $x^4-14x^2+x+38=0......(3)$.

As in many other cases of this nature, resort must be had to special expedients. *Vide Ray's New Higher Algebra*, sec. 253.

Equation (3) may be put in the form

$$x^4-2x^3+2x^3-4x^2-10x^2+20x-19x+38=0$$

by subtracting and adding $2x^3$, separating $-14x^2$ into $-4x^2$ and $-10x^2$, and placing $20x-19x$ for x.

$$x^3(x-2)+2x^2(x-2)-10x(x-2)-19(x-2)=0.$$

$(x-2)(x^3+2x^2-10x-19)=0$, by factoring.

$\therefore$ $x-2=0......(a)$, and $x^3+2x^2-10x-19=0......(b)$.

Hence $x=2$; substituting this value of x in equation (1) or (2), $y=3$.

I cannot discover any special artifice by which the roots of equation (b) can be found by quadratics. By Sturm's Theorem, the three roots of equation (b) are found to lie between 3 and 4, -3 and -2, -1 and -2. By Horner's Method, they are found to be $x_1=3.1313125$, $x_2=-3.283185$, $x_3=-1.84813652$; the corresponding values of y are $y_1=-2.8051181$, $y_2=-3.779310$, $y_3=3.58442837$.

X. Solutions selected by SYLVESTER ROBINS, North Branch Depot, N. J.

(A). From (1), $y=7-x^2$; from (2), $y=\sqrt{(11-x)}$.

$\therefore$ $7-x^2=\sqrt{(11-x)}$. By squaring, $49-14x^2+x^4=11-x$.

$\therefore$ $x^4-10x^2=4x^2-x-38$.

Add $2x^3-19x=2x^3-19x$, and $x^4+2x^3-10x^2-19x=2x^3+4x^2-20x-38$.

Factoring $x(x^3+2x^2-10x-19)=2(x^3+2x^2-10x-19)$.

$\therefore$ $x=2$, $y=3$.

[W. L. Harvey in "Our Young Folks," *New York Tribune.*]

(B). (1) $x^2+y=11$. (3) $x^2-9=2-y=d$, by assumption.
(2) $x+y^2=7$. (4) $x-3=4-y^2=sd$, by assumption.

Since $x-3=sd$, $(x-3)/s=d$, and $x^2-9=d$, so $x^2-9=(x-3)/s=(x/s)-(3/s)$.

$x^2-(x/s)=9-(3/s)$.

$x^2-(x/s)+(1/4s^2)=9-(3/s)+(1/4s^2)$.

$x-(1/2s)=3-(1/2s)$.

$\therefore$ $x=3$, and $y=2$. [From *The School Day Visitor.*]

NOTE on No. 78, Algebra. Mr. Boorman misapprehends the "real point" of my comment. I will endeavor to make my meaning plain.

Given $x^2+xy=10$ and $y^2+xy=15$ to find the values of x and y. Adding these two equations and taking the square root, we have $x+y=\pm5$. Dividing the first equation by this, we have $x=\pm2$, and in the same manner, $y=\pm3$.

I have supposed that it is *settled* that an equation of the second degree has two roots and *no more*, especially when the equation can be solved by an equation of that degree; and that when an equation of a higher degree, which has more than two roots, is needlessly used in the solution, still the number of roots of the original equation is not thereby increased, and that all, save two, of the roots of the higher equation are not roots of the original equation.

Now, Mr. Boorman, in his solution, page 43 of last volume, by transposing xy to the other side of the original equations and then multiplying the two equations together, obtains an equation of the *fourth* degree. So I said in my Note (and say now) that his solution seems to me to involve the following course of reasoning: If $x^2=36$......(1), then $x^4=1296$..... (2), and as equation (2) has four roots, therefore equation (1) has four roots: a manifest error as it seems to me.

I did not "mistake him to mean" that the roots of $x^4=16$, are ±2 and ∓2; but in his solution of No. 78, he says there are eight roots, viz.: $x=\mp2$ and $y=\mp3$, and $x=\pm2$ and $y=\pm3$. But I said that ±2 and ∓2, *taken by themselves*, are precisely the same, and therefore that his *eight* roots are really only *four*, *i. e.* two pairs, two values of x and two values of y, precisely as are obtained without transforming the original equations of the second degree into a biquadrate equation. JOSIAH H. DRUMMOND.

GEOMETRY.

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105. Proposed by B. F. FINKEL, A.M.,M.Sc.,Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Sind A, B, C, D vier harmonische Punkte und beschreibt man über dem Durchmesser AC einen Kreis, von welchem S ein beliebiger Punkt ist, so wird derjenige Kreisbogen, welcher innerhalb des Winkels BSD liegt, entweder von A oder von C halbirt. [*Reye's Geometrie der Lage*, page 191.]

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Sind A, B, C, D vier harmonische Punkte und S ein beliebigen Punkt auf dem Kreis ASC, so sind SA, SB, SC, und SD vier harmonische Strahlen, denn vier harmonische Punkte werden aus jedem Punkte (S) durch vier harmonische Strahlen projicirt.

Weil aber die Strahle SC auf der Strahle SA senkrecht steht, so halbirt sie den Winkel zwischen den anderen beiden, Strahlen SB und SD, und folglich den Kreis-bogen zwischen denselben, nach dem Satz: "Wenn von vier harmonischen Strahlen zwei getrennte auf einander senkricht stehen so halbiren sie die Winkel zwischen den anderen beiden Strahlen."

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II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let A, B, C, D be the given points.

Join AS, BS, CE, DS. Then $AB:BC=AD:DC$, or $AB:AD=BC:DC$.

$\therefore \triangle ASB:\triangle BSC=\triangle ASD:\triangle CSD$.

$\therefore AS.BS\sin ASB:BS.CS\sin BSC=AS.DS\sin ASD:CS.DS\sin DSC$.

$\therefore \dfrac{\sin ASB}{\sin BSC}=\dfrac{\sin ASD}{\sin DSC}$.

But $\sin ASB=\sin(\frac{1}{2}\pi-BSC)=\cos BSC$.

$\sin ASD=\sin(\frac{1}{2}\pi+DSC)=\cos DSC$.

$\therefore \cot BSC=\cot DSC$.

$\therefore \angle BSC=\angle DSC$.

$\therefore$ C bisects arc FCE. Similarly A bisects arc $EASF$.

CALCULUS.

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80. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A vessel is anchored in three fathoms of water, and the cable passes over a sheave in the bowsprit, which is six feet above the water. If the cable is hauled in at the rate of one foot a second, how fast is the vessel moving through the water when there is five fathoms of cable out? What is the acceleration of the vessel's velocity? [From *Byerly's Problems in Differential Calculus.*] ANS.—(a) 5-6 feet per second; (b) 12-121 feet per second. Are these results correct?

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio

Let x=the horizontal distance from sheave to anchor, and y=the length of cable out, both in feet.

Then $x=(y^2-576)^{\frac{1}{2}}$. Differentiating and making $y=30$ and $dy=-1$, we find $dx=-\frac{5}{3}$; that is, x *diminishes* at the rate of $\frac{5}{3}$ ft. per second, or the ship moves at the same rate. Differentiating again and making the same substitutions we get $d^2x=-\frac{8}{81}$, that is, the vessel's velocity is increasing at the rate $\frac{8}{81}$ ft. per second.

Accordingly the results given as answers to the problem are not correct. It is evident without a solution that the vessel's velocity is always *greater* than the rate at which the cable is hauled in.

II. Solution by ELMER SCHUYLER, High Bridge, N. J.

If I understand the problem, it is (1) $y^2=x^2+24^2$.

We want to find $dx/dt=v$, and $d^2x/dt^2=\alpha$. (2) $y/t=1$.

$\therefore y=t$; $dy=dt$.

$dy/dt=1$, *i. e.* rate is constant and has no acceleration.

$$(3)\ 2x\frac{dx}{dt}=2y\frac{dy}{dt}.$$

$$(4)\ \frac{dx}{dt}=\frac{y}{x}\cdot\frac{dy}{dt}=\frac{30}{\sqrt{30^2-24^2}}\times1=\tfrac{5}{3}\text{ ft.}\quad\text{1st answer.}$$

$$(5)\ \frac{d^2x}{dt^2}=\frac{x(dy/dt)(=\text{constant }1)-y(dx/dt)}{x^2}=\frac{x-(y^2/x)}{x^2}=\frac{x^2-y^2}{x^3}$$

$$=-\frac{24^2}{x^3}=-\frac{24^2}{18^3}=-\tfrac{8}{81}\text{ acceleration per second.}$$

III. **Results by J. SCHEFFER, A. M., Hagerstown, Md.**

If I understand the problem correctly, I find the values to be $\frac{5}{3}$ and $\frac{8}{81}$ instead of $\frac{5}{6}$ and $\frac{12}{121}$.

[NOTE.—A letter from Dr. Byerly states that this, as a number of other errors, crept into the work by oversight. EDITOR.]

MECHANICS.

63. **Proposed by A. H. BELL, Hillsboro, Ill.**

From a horizontal support at a distance of 10 feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

Comment by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Soon after my solution of the above problem was published Mr. George Richards and Mr. A. H. Bell called my attention to the incorrectness of my assumption in regard to the angles. The error was due to carelessness or inability on my part, either of which is inexcusable. Mr. A. H. Bell sent me the following admirable solution, and I doubt whether a simpler one can be effected. The fact that Mr. Bell has solved this problem is a sure guarantee of its correctness and accuracy.

Let $AB=c=10$, $AD=3=d$, $DC=g=5$, $BC=e=5$, $EB=x+y$, $AE=x-y$.

$$\cos E=\frac{(x+y)^2+(x-y)^2-c^2}{2(x+y)(x-y)}=\frac{(x+y-e)^2+(x-y-d)^2-g^2}{2(x+y-e)(x-y-d)}\ \ldots\ldots\ldots\ldots\ (1).$$

$$DF=CN=\frac{c^2+(x-y)^2-(x+y)^2}{2c(x-y)}(x-y-d)=\frac{c^2+(x+y)^2-(x-y)^2}{2c(x+y)}(x+y-e).(2).$$

$$e+d=a=8,\ e-d=b=2.$$

$$\therefore\ (1)=(4by+g^2-c^2-b^2)x^2-a(4y^2-c^2)x+(c^2+a^2-g^2)y^2-bc^2y$$

$$-\tfrac{1}{4}c^2(a^2-b^2)=0,\text{ or }x^2-\frac{32(y^2-25)}{8y-79}+\frac{139y^2-200y-1500}{8y-79}=0\ldots(3).$$

$$(2)=x^3-\tfrac{1}{2}ax^2-[y^2-\tfrac{1}{2}by+(bc^2/8y)]x+\tfrac{1}{8}ac^2=0,$$

$$\text{or } x^3-4x^2-[y^2-y+(25/y)]x+100=0 \quad \ldots\ldots(4).$$

Eliminating x between (3) and (4) we get,

$$1281024 00y^{12}-536601600y^{11}-9985725784y^{10}+36190002752y^9$$
$$+307839235264y^8-1004805985048y^7-4555231759005y^6$$
$$-4948451989304y^5+10854929220095 0y^4+26216813125200y^3$$
$$-54537984439125y^2+268623315000y-49234858937500=0 \quad \ldots(5).$$

The prodigious amount of work necessary to arrive at (5) is wonderful, and great credit is due Mr. Bell for the above solution.

DIOPHANTINE ANALYSIS.

72. **Proposed by H. C. WILKES, Skull Run, W. Va.**

Given $x^2+y^2+z^2=p^2+q^2+r^2$, to find unequal integral values for x, y, z, p, q, and r.

I. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

The conditions of the problem are satisfied in the following six identities, in which m, n and r represent any integers, the values being so chosen as to avoid *zero* in the residual quantities,

$$(m+n+r)^2+(m-r)^2+(n-r)^2=(m+n-r)^2+(m+r)^2+(n+r)^2,$$
$$(m+n+r)^2+(m-n)^2+(n-r)^2=(m-n+r)^2+(m+n)^2+(n+r)^2,$$
$$(m+n+r)^2+(m-n)^2+(m-r)^2=(m-n-r)^2+(m+n)^2+(m+r)^2,$$
$$(m+n-r)^2+(m-n)^2+(m+r)^2=(m-n+r)^2+(m+n)^2+(m-r)^2,$$
$$(m+n-r)^2+(m-n)^2+(n+r)^2=(m-n-r)^2+(m+n)^2+(n-r)^2,$$
$$(m-n+r)^2+(m-r)^2+(n+r)^2=(m-n-r)^2+(m+r)^2+(n-r)^2.$$

These equations can be reduced to the following two formulas :

$$\left(\frac{p+2q+2r}{3}\right)^2+\left(\frac{2p+q-2r}{3}\right)^2+\left(\frac{2p-2q+r}{3}\right)^2=p^2+q^2+r^2 \quad \ldots\ldots\ldots\ldots(1).$$

$$\left(\frac{2q-p+2r}{3}\right)^2+\left(\frac{2p+2q-r}{3}\right)^2+\left(\frac{2p-q+2r}{3}\right)^2=p^2+q^2+r^2 \quad \ldots\ldots\ldots\ldots(2).$$

To insure integral results, assign to p, q, and r multiples of 3.

As 3 things can be arranged in 6 different ways, there may be made, in regard to p, q, and r, 6 different substitutions with each set of assigned values.

In Formula (1), these substitutions will produce 3 different sets of values

$(2)=x^3-\frac{1}{2}ax^2-[y^2-\frac{1}{2}by+(bc^2/8y)]x+\frac{1}{8}ac^2=0$,

$$\text{or } x^3-4x^2-[y^2-y+(25/y)]x+100=0 \ldots\ldots (4).$$

Eliminating x between (3) and (4) we get,

$$\begin{aligned}
&1281024000y^{12}-5366016000y^{11}-99857257784y^{10}+361900027524y^{9}\\
&\quad+3078392352644y^{8}-10048059850489y^{7}-45552317590056y^{6}\\
&\qquad-49484519893044y^{5}+108549292200950y^{4}+262168131252004y^{3}\\
&\qquad\quad-545379844391250y^{2}+268623315000y-492348589375000=0 \quad \ldots(5).
\end{aligned}$$

The prodigious amount of work necessary to arrive at (5) is wonderful, and great credit is due Mr. Bell for the above solution.

DIOPHANTINE ANALYSIS.

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$$(m+n+r)^2+(m-n)^2+(m-r)^2=(m-n-r)^2+(m+n)^2+(m+r)^2,$$

$$(m+n-r)^2+(m-n)^2+(m+r)^2=(m-n+r)^2+(m+n)^2+(m-r)^2,$$

$$(m+n-r)^2+(m-n)^2+(n+r)^2=(m-n-r)^2+(m+n)^2+(n-r)^2,$$

$$(m-n+r)^2+(m-r)^2+(n+r)^2=(m-n-r)^2+(m+r)^2+(n-r)^2.$$

These equations can be reduced to the following two formulas:

$$\left(\frac{p+2q+2r}{3}\right)^2+\left(\frac{2p+q-2r}{3}\right)^2+\left(\frac{2p-2q+r}{3}\right)^2=p^2+q^2+r^2 \;\ldots\ldots\ldots\ldots (1).$$

$$\left(\frac{2q-p+2r}{3}\right)^2+\left(\frac{2p+2q-r}{3}\right)^2+\left(\frac{2p-q+2r}{3}\right)^2=p^2+q^2+r^2 \;\ldots\ldots\ldots\ldots (2).$$

To insure integral results, assign to p, q, and r multiples of 3.

As 3 things can be arranged in 6 different ways, there may be made, in regard to p, q, and r, 6 different substitutions with each set of assigned values.

In Formula (1), these substitutions will produce 3 different sets of values

when the assigned values are different numbers. In Formula (2), however, only one new set will be obtained for each set of assigned values.

To obtain directly the 3 sets of values by Formula (1), arrange the set of assigned values in the order of their numerical greatness beginning with the largest number; then substitute these respectively for p, q, r; q, p, r; r, q, p.

II. Solution by CHARLES C. CROSS, Libertytown, Md.

$(m-n+p+q)^2+(m-p-q)^2+(m+n)^2=(m+n-p-q)^2+(m+p+q)^2+(m-n)^2$,

$(m+n-p+q)^2+(m-n-q)^2+(m+p)^2=(m-n+p-q)^2+(m+n+q)^2+(m-p)^2$,

$(m+n+p-q)^2+(m-n-p)^2+(m+q)^2=(m-n-p+q)^2+(m+n+p)^2+(m-q)^2$.

Three sets of the sum of three squares equals the sum of three other squares, having no term in common.

Let $m=25$, $n=10$, $p=3$ and $q=1$.

Then $19^2+21^2+35^2=31^2+29^2+15^2$; $33^2+14^2+28^2=17^2+36^2+22^2$, and $37^2+12^2+26^2=13^2+38^2+24^2$.

Three other sets of the sum of three squares equal to the sum of three other squares are given below, but they each have two terms in common.

$(m+n+p+q)^2+(m-n-p)^2+(m-q)^2=(m-n-p-q)^2+(m+n+p)^2+(m+q)^2$,

$(m+n+p+q)^2+(m-p-q)^2+(m-n)^2=(m-n-p-q)^2+(m+p+q)^2+(m+n)^2$,

$(m+n+p+q)^2+(m-n-q)^2+(m-p)^2=(m-n-p-q)^2+(m+n+q)^2+(m+p)^2$.

If we let $m=am$, $n=bn$, $p=cp$, and $q=dq$, the above sets become very general.

III. Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

It is manifest that if we find more than one integral value for x, y, and z, in $x^2+y^2+z^2=a$ the question is solved.

Take $x^2+y^2+z^2=p^2$; assume $z^2=2xy$ and reducing we have $x+y=p$, and $y=p-x$, and $z^2=2px-2x^2=\square=$(say) q^2x^2. Then

$$x=\frac{2p}{q^2+2},\ y=\frac{q^2p}{q^2+2},\ \text{and } z=\frac{qp}{q^2+2}$$

in which q may be any number. Taking $q=1$, 2, 3, etc., we obtain values of x, y, and z in terms of p, and any number of them that we choose. To obtain integral values, take $p=$the greatest common multiple of the denominators of the values taken, and we have as many values of x, y, and z, the sum of whose squares is constant; and of course the sum of the squares of each set equals the sum of the squares of every other set. For example, take $q=1$, 2, 3, 4.

$$x=\frac{2p}{3},\ \frac{2p}{6},\ \frac{2p}{11},\ \frac{2p}{18}.$$

$$y=\frac{p}{3},\ \frac{4p}{6},\ \frac{9p}{11},\ \frac{16p}{18}.$$

$$z=\frac{2p}{3},\quad \frac{4p}{6},\quad \frac{6p}{11},\quad \frac{8p}{18}.$$

Take $p=198$.

$x=132$	66	36	22.
$y=$ 66	132	162	176.
$z=132$	132	108	88.

Rejecting the first or second values, we have three solutions of the question. Of course, we may take any two different sets of values and obtain one solution of the question.

IV. **Solution by A. H. BELL, Hillsboro, Ill.**

Let $x^2+y^2+z^2=p^2+q^2+r^2=(a^2+b^2+e^2)(c^2+d^2)$.

$(c^2+d^2)=\square$.

Then the general value will become $(ac\pm bd)^2+(bc\mp ad)^2+e^2(c^2+d^2)$.

By substitution we obtain 7 sets of values. When paired there are 12 sets of unequal values answering the requirements of this problem.

V. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.**

Let $x=6m$, $y=10m$, $z=11m$, $p=7m$, $q=8m$, $r=12m$.

$\therefore\ x^2+y^2+z^2=p^2+q^2+r^2$.

$\therefore\ (6m)^2+(10m)^2+(11m)^2=(7m)^2+(8m)^2+(12m)^2=257m^2$, where m can have any integral value.

VI. **Solution by the PROPOSER.**

Let $x=m^2+n^2$, $y=m^2+mn-n^2$, $z=m^2-mn-n^2$.

We have $\overset{x^2}{(m^2+n^2)^2}+\overset{y^2}{(m^2+mn-n^2)^2}-\overset{z^2}{(m^2-mn-n^2)^2}=(m^2+2mn-n^2)^2=p^2$.

$$\overset{x^2}{(m^2+n^2)^2}+\overset{z^2}{(m^2-mn-n^2)^2}-\overset{y^2}{(m^2+mn-n^2)^2}=(m^2-2mn-n^2)^2=q^2.$$

$$\overset{z^2}{(m^2-mn+n^2)^2}+\overset{y^2}{(m^2+mn+n^2)}-\overset{x^2}{(m^2+n^2)^2}=(m^4-4m^2n^2+n^4)=z^2r^2.$$

Hence to find integral numbers that will fit the equation, we must find integral values for m and n that will make $(m^4-4m^2n^2+n^4)$ a square.

Let $m^4-4m^2n^2+n^4=r^2$ ....(1).

$m^4-4m^2n^2+4n^2=(r+a)^2$ ....(2).

$m^4-2m^2n^2+n^4=(r+b)^2$ ....(3).

From (1)+(2) we have $r=\dfrac{3n^4-a^4}{2a}$ ......(4), $m^2=2n^2+\dfrac{3n^4+a^2}{2a}$ ....(5).

From (1)+(3) we have $r=\dfrac{2m^2n^2-b^2}{2b}$ ...(6), $m^2=n^2+\dfrac{2m^2n^2+b^2}{2b}$ ...(7).

From (4)+(6), $2am^2n^2-3bn^4=ab^2+a^2b$ ....(8).

From (5)+(7), $2am^2n^2-9bn^4-2abn^2=ab^2+a^2b$ ....(9).

Subtracting (9) from (8), $2abn^2=2ab(b-a)$. Whence $b-a=n^2$.

Factoring (8), $n^2(2am^2-3bn^2)=ab(b-a)$. Whence $2am^2-3bn^2=ab$.

Put $2a=n$ and by easy reduction we have $2m^2=b(6n+1)$. Since $\frac{1}{2}(6n+1)$ cannot be a square $b/2$ must be a square to make an integral. Then $m^2=b/2(6n+1)$ is $(n^2+\frac{1}{2}n)/2\times(6n+1)$. $n=4$ being the only value that will make both factors a square, $m=\sqrt{[(n^2+\frac{1}{2}n)/2]}\times\sqrt{(6n+1)}=15$.

$\therefore$ $x=241$, $y=269$, $z=149$, $p=329$, $q=89$, $r=191$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

106. Proposed by ELMER SCHUYLER, High Bridge. N. J.

What is the amount of \$1000 at compound interest for 3 years at 6%, if it be compounded every instant?

107. Proposed by R. V. ALLEN, Hooker Station, Ohio.

A barn, $ABCD$, length $AB=b$ feet, width $AD=a$ feet, standing in an open field, has a horse tethered to a point, P, in the side, AB, distance $AP=c$ feet, with a rope R feet long. Over what area can the horse graze?

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 10.

ALGEBRA.

94. Proposed by J. W. YOUNG, Columbus, Ohio.

Solve: $\left[\dfrac{x^2+14x+1}{p^4+14p^2+1}\right]^3=\dfrac{x(x-1)^4}{p^2(p^2-1)^4}$.

Burnside and Panton's *Theory of Equations*, page 148, ex. 17.

95. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Substitute *numbers* in place of the letters in the following pattern: ...... $\triangle=\sqrt{81^2a^2b^2c^2}=81abc$..... b^2+c^2, a^2+c^2, a^2+b^2; and compute the areas and sides of the whole nest of integral, rational triangles.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

GEOMETRY.

114. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

If a variable ellipse hyperosculate a fixed ellipse at the extremity of the minor axis, the locus of the foci is a circle whose diameter is equal to the radius of curvature.

115. Proposed by MARY BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

The locus of a point such that the sum of the squares of its normals form a given ellipsoid is constant, is a co-axial ellipsoid. [From *C. Smith's Solid Analytical Geometry*, page 95.]

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CALCULUS.

85. Proposed by J. SCHEFFER. A. M., Hagerstown, Md.

A line of double curvature, beginning at some point in the circumference of the base circle of a right cone, winds itself under the constant inclination β to the base circle around the curved surface of the cone. Find its length and that of its projection upon the base circle.

86. Proposed by WALTER H. DRANE. Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove that the curve whose normal equals its radius of curvature drawn in an opposite direction, is the catenary, $y=c\cosh(x/c)$.

87. Proposed by MARY BLAINE. B. Sc., Graduate Student, Drury College, Springfield, Mo.

Integrate $(px-y)(py+x)=h^2p$, where $p=dy/dx$.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

MECHANICS.

82. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street. Cambridge, Mass.

A sphere, diameter $2a$, rests in limiting equilibrium upon the edge of a box and against a vertical wall. If the box be of such dimensions that it will not tip, find the distance of the box from the wall, having given the coefficient of friction between the sphere and wall $\frac{1}{2}$, between the sphere and box $\frac{1}{3}$, and between the box and floor $\frac{2}{3}$. [From Problems in Mechanics proposed to class in Harvard University.]

83. Proposed by MARY BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

A particle is projected upwards in vacuo with a velocity v. Show that on reaching the ground again there is no deviation to the south, but the deviation to the west is $4\omega\cos\lambda(v^3/3g^2)$. [*Laplace*, iv. page 341.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 10.

DIOPHANTINE ANALYSIS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Find three square numbers in harmonical progression.

79. Proposed by EDMUND FISH, Hillsboro, Ill.

Find an integral right triangle in which the bisector of one of the acute angles is also integral.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

MISCELLANEOUS.

72. **Proposed by DR. E. D. ROE, JR., Associate Professor of Mathematics, Oberlin College, Oberlin, Ohio.**

If a, b, and c are integers, and

$$\left.\begin{array}{l} b,\ c-b,\ c-1 \\ c-a-1 \\ c-a-1 \end{array}\right\} > 0,$$

$$c-a-b-1 \geqq 0,$$

prove that the sum of the series,

$$1+\frac{a.b}{1.c}+\frac{a(a+b).b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}+\ldots\ldots,$$

is equal to

$$\frac{(c-1)\,!\,(c-a-b-1)\,!}{(c-a-1)\,!\,(c-b-1)\,!}$$

73. **Proposed by CHARLES E. MYERS, Canton, Ohio.**

In an ice cream freezer, cream of a homogeneous character and at the uniform temperature of 60° Fahrenheit is put into a cylinder having a closed base, and the whole put into a freezing mixture so as to subject the base and convex surface to a constant temperature of 30° Fahrenheit. Required the temperature at any point within the cream after the expiration of a given time. [From *Higher Mathematics.*]

74. **Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.**

The longest diameter of a horizontal ellipse is $CB=2a=6$ feet. Its shortest diameter is $EF=2b=4$ feet, their intersection being at D. Find in an indefinite vertical plane passing through CB, a point A 5 feet$=c$ from D, the ellipse being seen from A as a circle.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 10.

EDITORIALS.

Dr. Robert J. Aley, of the University of Indiana, has been elected to membership in the *Deutsche Mathematiker-Vereinigung*, and also in the *London Mathematical Society.*

This issue has been somewhat delayed by the illness of the editor. We shall make strenuous efforts to have all subsequent numbers reach our subscribers by the last of each month.

We are following our previous plan of sending out the January number to each of our old subscribers. Any one wishing to discontinue should return this number with his name and address legibly written on the wrapper.

Contributors should observe the following in sending in contributions: 1. Write only on one side of the paper; 2. Sign your name and address to each contribution: 3. In contributing problems or solutions, sign your name to

each problem or solution, and let each problem or solution be on a separate sheet of paper. By observing these directions, your contributions will often be saved from the waste basket.

BOOKS AND PERIODICALS.

A Text-Book of Statics. By William Briggs, M. A., LL. B., F. R. A. S., General Editor of the Tutorial Series, Principal of University Correspondence College ; and G. H. Bryan, M. A., Smith's Prizeman, Fellow St. Peter's College, Cambridge. 8vo. Cloth, 220 pages. Price, 3*s.* 6*d.* New York : Hinds & Noble, Publishers.

This little work presents in a very excellent manner all the facts and principles in Statics that can be mastered by a student in under graduate work. All principles are clearly presented and illustrated by excellent diagrams. Many illustrative problems are solved and many problems are inserted at the end of each chapter. B. F. F.

Elements of Trigonometry, Plane and Spherical. By Andrew W. Phillips, Ph. D., and Wendell M. Strong, Ph. D., Yale University. 8vo. Cloth, 138 pages. Price, 90 cents. New York : Harper & Brothers.

This work, in addition to the usual matter treated, contains several features which are entirely distinctive: First, in Spherical Trigonometry, we have the photographic reproduction of models used in Yale University. These are beautiful in themselves and add a charm to the book only equaled by that of the Elements of Geometry by Professors Phillips and Fisher. Second, the graphic representation of the Trigonometric and Inverse Trigonometric Functions. Third, the brief treatment of Plane, Spherical and Pseudo-Spherical Trigonometries. In this treatment it is pointed out that Plane Trigonometry is a special case of Spherical Trigonometry, or better, is the limiting case of Spherical Trigonometry and Pseudo-Spherical Trigonometry. This discussion might have been enlarged upon to advantage, but not, however, without the use of some principles of the Calculus.

The printed page does not have the artistic appearance which the work deserves, it presenting the appearance of being printed from old type. B. F. F.

On the Study and Difficulties of Mathematics. By Augustus De Morgan. New Edition. 8vo. Cloth. 288 pages. Price. $1.00. Chicago : The Open Court Publishing Co.

This work is of special interest to all teachers of students of mathematics. In it are explained all the difficulties that arise in the study of elementary mathematics, making it possible for the student to master the first principles of mathematics in a way that will make the study of higher mathematics a joy forever, in that he will not need to continually return to unlearn principles taught him in the elements. The chapter on Arithmetical Fractions should be read several times, as it makes very clear what has been a necessity in the development and progress of mathematics, viz., the carrying of terms in that which is simple to that which is complex, and enlarging their meaning as our ideas enlarge. A failure to grasp this important principle has led to endless and useless discussions; as, for example, whether 3 and $3\sqrt{-1}$ are numbers.

The whole work bears the impress of its author's genius. It is a notable instance of a mathematician of eminent mathematical attainment setting himself the task of ridding

the elementary branches of mathematics of some of their inaccuracies and solecisms. Great credit is due the editor, Mr. T. J. McCormack. for his service to the cause of mathematics in bringing out new editions of such valuable works. B. F. F.

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ON SYMMETRIC FUNCTIONS.

By DR. E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from January Number.]

This term is ever found in

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} p_{i_1} p_{i_2} p_{i_3}$$

where p_{i_1} is to be applied to the j_1st line, p_{i_2} to the j_2d line, and p_{i_3} to the j_3d line. The term then corresponds to the substitution $\begin{pmatrix} j_1 & j_2 & j_3 \\ 1 & 2 & 3 \end{pmatrix}$, and has the same sign as before. We are now ready to state the general case.

g. We must show that it is indifferent whether we apply the exponents in all possible permutations to the columns or to the rows of D, *i. e.*, that to every term of the second formation corresponds the same term in the first formation, and then since the second formation is the straightforward product of D and Σ, it will follow that the first formation is equal to the product of D and Σ. We may denote the second and first formations by

$$\begin{vmatrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \ldots\ldots \alpha_1^{n-1} \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \ldots\ldots \alpha_2^{n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_n^0 & \alpha_n & \alpha_n^2 \ldots\ldots \alpha_n^{n-1} \end{vmatrix} \begin{matrix} p_{i_1} \\ p_{i_2} \\ \vdots \\ p_{i_n} \end{matrix} \text{ and } \begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \ldots\ldots \alpha_n^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \ldots\ldots \alpha_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} \ldots\ldots \alpha_n \end{vmatrix} p_{i_1} p_{i_2} \cdot\cdot p_{i_n}.$$

$i_1, i_2, \ldots\ldots i_n$ form a permutation of the numbers 1, 2, 3, ... n, and the expression on the left signifies that D has been multiplied by $\alpha_1{}^{p_{i_1}}\alpha_2{}^{p_{i_2}}\alpha_n{}^{p_{i_n}}$ of $\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}$, the first row by $\alpha_1{}^{p_{i_1}}$, the second by $\alpha_2{}^{p_{i_2}}$, ... the nth by $\alpha_n{}^{p_{i_n}}$, or more briefly that p_{i_1} is to be applied as exponent to the elements of the first row, p_{i_2} in the same way to those of the second, ... and p_{i_n} similarly to those of the nth row, while the expression on the right (D with columns changed into rows and horizontal line of p's) shall signify that the p's are to be applied arbitrarily as exponents to the lines of D as written, the same thing as applying them to the columns as before written, and in such order that p_{i_1} is applied to the j_1st line, p_{i_2} to the j_2d line, ...p_{i_n} to the j_nth line.

h. Any term $\alpha_1{}^{p_{i_1}+j_1-1}\alpha_2{}^{p_{i_2}+j_2-1}\alpha_3{}^{p_{i_3}+j_3-1}\ldots\ldots\alpha_n{}^{p_{i_n}+j_n-1}$ (an expression which is seen to be an $(n\,!)^2$ valued function when one permutes the n p's in all possible ways, and also the n j's) correspond, with reference to the expression on the left hand, to the $(n\,!)^2$ valued substitution

$$s_1 s_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ j_1 & j_2 & j_3 & \ldots\ldots & j_n \end{pmatrix}$$

the first factor referring to the vertical series of p's, the second afterwards to the determinant, when the indicated multiplications have been performed. With reference to the right hand expression, corresponds the $(n\,!)^2$ valued substitution

$$s_1{}^1 s_2{}^1 = \begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ 1 & 2 & 3 & \ldots\ldots & n \end{pmatrix},$$

in a similar manner, the first factor referring to the horizontal series of p's, and the second to the determinant after the p's have been applied to the elements, to the same term. It is clear that, numerically at least, the term is the same, for we must apply p_{i_1} to the j_1st line, and then take the first column ; this gives $\alpha_1{}^{p_{i_1}+j_1-1}$; next we must apply p_{i_2} to the j_2d line, and take the second column ; this gives $\alpha_2{}^{p_{i_2}+j_2-1}$; in general we must apply p_{i_r} to the j_rth line, and then take the rth column ; this gives $\alpha_r{}^{p_{i_r}+j_r-1}$, and the product $\prod\limits_{r=1}^{r=n} \alpha_r{}^{p_{i_r}+j_r-1}$ is numerically the term in question. The first substitutions s_1 and $s_1{}^1$ in either case have no effect on the sign of the term ; they merely assign the p's to their proper lines. The second substitutions

$$s_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ j_1 & j_2 & j_3 & \ldots\ldots & j_n \end{pmatrix} \text{ and } s_2{}^1 = \begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ 1 & 2 & 3 & \ldots\ldots & n \end{pmatrix}$$

are reciprocals and have the same modulus or sign factor.

i. We have proved : The compound substitution $s_1 s_2$ corresponds to all the $(n\,!)^2$ terms which $D\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}$ is capable of having. Similarly the compound substitution $s_1{}^1 s_2{}^1$ which is simultaneous with $s_1 s_2$ and depends thereon corresponds to the $(n\,!)^2$ terms of $(p_1 p_2 \ldots\ldots p_n)$ and the terms corres-

ponding to $s_1^1 s_2^1$ are identical term by term with the terms corresponding to $s_1 s_2$. Therefore the theorem follows that

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=(p_1 p_2\ldots\ldots p_n)=\Sigma(p_{i_1}\ \ p_{i_2}+1\ \ p_{i_3}+2\ldots\ldots p_{i_n}+n-1),$$

where $i_1, i_2, \ldots\ldots i_n$ form a permutation of the numbers $1, 2, \ldots\ldots n$.

B. Another Method. Elimination by Means of Symmetric Functions.

The problem of eliminating the variable between two binary forms by means of symmetric functions requires the calculation of the latter, and thus leads to the demand for symmetric functions as a whole. The calculation of all eliminants or resultants in succession is therefore, from this standpoint, the systematic calculation of all symmetric functions. In other words, the problems of calculating all resultants and of calculating all symmetric functions are identical. Given all resultants, we may write down the values of all symmetric functions. Given all symmetric functions, we may write down the values of all resultants. This idea is fruitful in giving rise to the following method of solving both problems simultaneously, and in yielding symmetric functions as a whole. The method will be illustrated and explained by one or two earlier cases, from which it will be seen that it can be carried as far as one pleases.

1. Two Quadratic Forms.

(1). *The Resultant.*

The resultant of two forms $a_0x^2+a_1x+a_2$ and $b_0x^2+b_1x+b_2$ of the second degree is (cf. p. 4)

$$b_0^2(a_0\beta_1^2+a_1\beta_1+a_2)(a_0\beta_2^2+a_1\beta_2+a_2)=$$

$$a_0^2(b_0\alpha_1^2+b_1\alpha_1+b_2)(b_0\alpha_2^2+b_1\alpha_2+b_2)=$$

$$b_0^2(a_0^2\Sigma\beta_1^2\beta_2^2+a_0a_1\Sigma\beta_1^2\beta_2+a_0a_2\Sigma\beta_1^2+a_1^2\Sigma\beta_1\beta_2+a_1a_2\Sigma\beta_1+a_2^2)=$$

$$a_0^2(b_0^2\Sigma\alpha_1^2\alpha_2^2+b_0b_1\Sigma\alpha_1^2\alpha_2+b_0b_2\Sigma\alpha_1^2+b_1^2\Sigma\alpha_1\alpha_2+b_1b_2\Sigma\alpha_1+b_2^2).$$

(2). *Aronhold's Operator.*

Applying Aronhold's Operator $\delta=b_0D_{a_0}+b_1D_{a_1}+b_2D_{a_2}$ on the first form of the resultant, first, using $b_0D_{a_0}$, then $b_1D_{a_1}$, and then $b_2D_{a_2}$, and denoting the coefficient of a_ia_k in the resultant by $|\ a_ia_k\ | = b_0^2\Sigma\beta_1^{2-i}\beta_2^{2-k}$, we get,

$$2a_0b_0\ |\ a_0^2\ |\ +b_0a_1\ |\ a_0a_1\ |\ +b_0a_2\ |\ a_0a_2\ |$$

$$+a_0b_1\ |\ a_0a_1\ |\ +2a_1b_1\ |\ a_1^2\ |\ +b_1a_2\ |\ a_1a_2\ |$$

$$+a_0b_2\ |\ a_0a_2\ |\ +a_1b_2\ |\ a_1a_2\ |\ +2a_2b_2\ |\ a_2^2\ |\equiv 0.$$

(3). *Identical Relations between Symmetric Functions.*

Since the expressions within the vertical strokes are functions of the b's,

and we can factor by columns, and take out the factors a_0, a_1, a_2 and since, for the rest, the whole expression is zero, whatever the values of the a's, it follows that their coefficients are zero, and identically zero, for they are zero for all values of the b's. We thus get :

$$2b_0 \mid a_0^2 \mid + b_1 \mid a_0 a_1 \mid + b_2 \mid a_0 a_2 \mid \equiv 0, \text{ coefficient of } a_0,$$

$$b_0 \mid a_0 a_1 \mid + 2b_1 \mid a_1^2 \mid + b_2 \mid a_1 a_2 \mid \equiv 0, \text{ coefficient of } a_1,$$

$$b_0 \mid a_0 a_2 \mid + b_1 \mid a_1 a_2 \mid + 2b_2 \mid a_2^2 \mid \equiv 0, \text{ coefficient of } a_2.$$

From these identities also follows :

$$\begin{vmatrix} 2\mid a_0^2 \mid & \mid a_0 a_1 \mid & \mid a_0 a_2 \mid \\ \mid a_1 a_0 \mid & 2\mid a_1^2 \mid & \mid a_1 a_2 \mid \\ \mid a_2 a_0 \mid & \mid a_2 a_1 \mid & 2\mid a_2^2 \mid \end{vmatrix} \equiv 0,$$

an identical relation between the six symmetric functions of a quadratic form which enter into the resultant, with another quadratic form. It is seen to be a symmetric determinant.

(4). *Application of Relations to Calculate Symmetric Functions.*

We may use the preceding identities to find the symmetric functions involved, of which it may be taken for granted that we know

$$\mid a_0^2 \mid, \mid a_1^2 \mid, \mid a_2^2 \mid, \text{ and } \mid a_1 a_2 \mid, \text{ or } b_0^2 \Sigma(\beta_1 \beta_2)^2, b_0^2 \Sigma \beta_1 \beta_2, b_0^2 \Sigma(\beta_1 \beta_2)^0,$$

$$\text{and } b_0^2 \Sigma \beta_1 \text{ equal to } b_2^2,\ b_0 b_2,\ b_0^2, \text{ and } -b_0 b_1.$$

Using these values with the first two identities, we have :

$$b_1 \mid a_0 a_1 \mid + b_2 \mid a_0 a_2 \mid = -2b_0 b_2^2$$

$$b_0 \mid a_0 a_1 \mid + 0 \mid a_0 a_2 \mid = -b_0 b_1 b_2.$$

Of these the second gives $\mid a_0 a_1 \mid = -b_1 b_2$, and by substituting this value in the first, $\mid a_0 a_2 \mid = -2b_0 b_2 + b_1^2$, the same results as appear in the table on page 5, when fx is changed into ϕx.

2. Two Cubic Forms.

We will next obtain relations between the symmetric functions which occur in the resultant of two forms of the third degree.

(1). *The Resultant.*

The resultant is equal to

$$a_0^3 (b_0 \alpha_1^3 + b_1 \alpha_1^2 + b_2 \alpha_1 + b_3)(b_0 \alpha_2^3 + b_1 \alpha_2^2 + b_2 \alpha_2 + b_3)$$

$$\times (b_0 \alpha_3^3 + b_1 \alpha_3^2 + b_2 \alpha_3 + b_3) = (-1)^{3 \times 3}$$

$b_0^3(a_0\beta_1^3+a_1\beta_1^2+a_2\beta_1+a_3)(a_0\beta_2^3+a_1\beta_2^2+a_2\beta_2+a_3)(a_0\beta_3^3+a_1\beta_3^2+a_2\beta_3+a_3)$

$=-(a_0^3 \mid 0^3 \mid +a_0^2a_1 \mid 0^21 \mid +a_0^2a_2 \mid 0^22 \mid +a_0^2a_3 \mid 0^23 \mid +a_0a_1^2 \mid 01^2 \mid$

$+a_0a_2^2 \mid 02^2+a_0a_3^2 \mid 03^2 \mid +a_1^3 \mid 1^3 \mid +a_1^2a_2 \mid 1^22 \mid +a_1^2a_3 \mid 1^23 \mid$

$+a_1a_2^2 \mid 12^2 \mid +a_1a_3^2 \mid 13^2 \mid +a_2^3 \mid 2^3 \mid +a_2^2a_3 \mid 2^23 \mid +a_2a_3^2 \mid 23^2 \mid +a_3^3 \mid$

$3^3 \mid +a_0a_1a_2 \mid 012 \mid +a_0a_1a_3 \mid 013 \mid +a_0a_2a_3 \mid 023 \mid +a_1a_2a_3 \mid 123 \mid)$*.

(2). *Aronhold's Operator and the Identical Relations.*
Applying Aronhold's operators and collecting the coefficients of

$$a_0^2,\ a_1^2,\ a_2^2,\ a_3^2,\ a_0a_1,\ a_0a_2,\ a_0a_3,\ a_1a_2,\ a_1a_3,\ a_2a_3,$$

we have, putting each equal to zero :

$3b_0 \mid 0^3 \mid +b_1 \mid 0^21 \mid +b_2 \mid 0^22 \mid +b_3 \mid 0^23 \mid \equiv 0$, coefficient of a_0^2,

$b_0 \mid 01^2 \mid +3b_1 \mid 1^3 \mid +b_2 \mid 1^22 \mid +b_3 \mid 1^23 \mid \equiv 0$, coefficient of a_1^2,

$b_0 \mid 02^2 \mid +b_1 \mid 12^2 \mid +3b_2 \mid 2^3 \mid +b_3 \mid 2^23 \mid \equiv 0$, coefficient of a_2^2,

$b_0 \mid 03^2 \mid +b_1 \mid 13^2 \mid +b_2 \mid 23^2 \mid +3b_3 \mid 3^3 \mid \equiv 0$, coefficient of a_3^2,

$2b_0 \mid 0^21 \mid +2b_1 \mid 01^2 \mid +b_2 \mid 012 \mid +b_3 \mid 013 \mid \equiv 0$, coefficient of a_0a_1,

$2b_0 \mid 0^22 \mid +b_1 \mid 012 \mid +2b_2 \mid 02^2 \mid +b_3 \mid 023 \mid \equiv 0$, coefficient of a_0a_2,

$2b_0 \mid 0^23 \mid +b_1 \mid 013 \mid +b_2 \mid 023 \mid +2b_3 \mid 03^2 \mid \equiv 0$, coefficient of a_0a_3,

$b_0 \mid 012 \mid +2b_1 \mid 1^22 \mid +2b_2 \mid 12^2 \mid +b_3 \mid 123 \mid \equiv 0$, coefficient of a_1a_2,

$b_0 \mid 013 \mid +2b_1 \mid 1^23 \mid +b_2 \mid 123 \mid +2b_3 \mid 13^2 \mid \equiv 0$, coefficient of a_1a_3,

$b_0 \mid 023 \mid +b_1 \mid 123 \mid 2b_2 \mid 2^23 \mid +2b_3 \mid 23^2 \mid \equiv 0$, coefficient of a_2a_3.

(3). *Analysis of the Operator into Three Operators.*
We may notice the formation of these equations. They contain three operators ; an operator 0, 1, 2, 3 applied to the indices of the a's whose coefficient we seek, gives the combinations within the strokes ; b_0, b_1, b_2, b_3 give the literal coefficients and when the exponent of one of the indices in the strokes exceeds the exponent of the same index of the a's, whose coefficient we seek, this exponent becomes the numerical coefficient of the term in question. *E. g.*, for the coefficient of a_2^2, we apply 0, 1, 2, 3, to $\mid 2^2 \mid$ and get

*The coefficient of $a_ia_ka_j$ has been farther abbreviated to $\mid i\ k\ j \mid$ for obvious reasons.

$|02^2|$, $|12^2|$, $|2^3|$, $|2^23|$ since here $|i\ k| = |k\ i|$.
b_0, b_1, b_2, b_3, are the corresponding literal coefficients ;
1, 1, 3, 1 are the corresponding numerical coefficients, and

$b_0|02^2| + b_1|12^2| + 3b_2|23^3| + b_3|2^23|$ is the coefficient of a_2^2. Again, for the coefficient of a_1a_3, 0, 1, 2, 3, to $|13|$ give

$|013|$, $|1^23|$, $|123|$, $|13^2|$ also we have
b_0, b_1, b_2, b_3, and
1, 2, 1, 2 ; finally

$b_0|013| + 2b_1|1^23| + b_2|123| + 2b_3|13^2|$ =the coefficient of a_1a_3.

(4). *Determinant Relations between Symmetric Functions.*

From the first four equations of (2) we have by elimination of b_0, b_1, b_2, b_3

$$\begin{vmatrix} 3|0^3| & |0^21| & |0^22| & |0^23| \\ |01^2| & 3|1^3| & |1^22| & |1^23| \\ |02^2| & |12^2| & 3|2^3| & |2^23| \\ |03^2| & |13^2| & |23^3| & 3|3^3| \end{vmatrix} = 0.$$

In a similar way, using four equations at a time of the ten, we should have $\dfrac{10.9.8.7}{1.2.3.4} = 210$ identically vanishing determinant relations between the symmetric functions involved.

(5). *The Calculation of the Functions by the Identical Equations.*

a. Of the stroked elements in the equations of (2) we know $|0^3|$, $|1^3|$, $|2^3|$, $|3^3|$, $|2^23|$, $|23^2|$. They are $b_0^3\Sigma(\beta_1\beta_2\beta_3)^3$, $b_0^3\Sigma(\beta_1\beta_2\beta_3)^2$, $b_0^3\Sigma(\beta_1\beta_2\beta_3)$, $b_0^3\Sigma(\beta_1\beta_2\beta_3)^0$, $b_0^3\Sigma\beta_1\beta_2$, $b_0^3\Sigma\beta_1$, equal to $-b_3^3$, $b_0b_3^2$, $-b_0^2b_3$, b_0^3, $b_0^2b_2$, $-b_0^2b_1$, respectively.

b. We may use them for solving the functions as follows : The equations of (2) become in order,

$$3b_0^4\Sigma(\beta_1\beta_2\beta_3)^3 + b_1b_0^2\Sigma(\beta_1\beta_2\beta_3)^2b_0\Sigma\beta_1\beta_2 + b_2b_0\Sigma\beta_1\beta_2\beta_3b_0^2\Sigma\beta_1^2\beta_2^2$$

$$+ b_0^3b_3\Sigma\beta_1^3\beta_2^3 = 0, \text{ or } -3b_0b_3^3 + b_1b_2b_3^2 - b_2^2b_0^2\Sigma\beta_1^2\beta_2^2 + b_3b_0^3\Sigma\beta_1^3\beta_2^3 = 0,$$

and in a similar way, $-b_0b_1b_3^2 + 3b_0b_1b_3^2 - b_0b_2^2b_3 + b_3b_0^3\Sigma\beta_1^2\beta_2^2 = 0$. These two give $b_0^3\Sigma\beta_1^2\beta_2^2$ and $b_0^2\Sigma\beta_1^2\beta_2^2$. The third equation gives

$$b_0b_0^3\Sigma\beta_1^3\beta_2\beta_3 + b_1b_0^3\Sigma\beta_1^2\beta_2\beta_3 + 3b_2b_0^3\beta_1\beta_2\beta_3 + b_3b_0^3\Sigma\beta_1\beta_2 = 0, \text{ or}$$

$$-b_0b_3b_0^2\Sigma\beta_1^2 + b_0b_1^2b_3 - 3b_0^2b_2b_3 + b_0^2b_2b_3 = 0.$$

It gives $\Sigma\beta_1^2$, but that is known if the previous resultant is calculated. The next equation gives $b_0^4\Sigma\beta_1^3 + b_1b_0^3\Sigma\beta_1^2 - b_0^2b_1b_2 + 3b_0^3b_3 = 0$.

With the preceding it gives $b_0^3\Sigma\beta_1^3$.

[To be continued.]

TEOREMA.

By J. M. MONSANTO, Mayaguez, Porto Rico, W. I.

Encontré en un libro, que el cubo de un número menos su raiz, es un multiplo de 6, es decir que $x^3-x=6n$, pero no lo demostraba. Hallé la demostracion y traté de sacar algun partido de esté teorema. Nada encontré, pero en mis investigaciones di con otro teorema que he aplicado a la extracion de la raiz cubica.

El teorema es el siguiente : Todo número dividida por 6, dá un residuo igual al que dá su cubo dividido por dicho número 6, o vice versa,—todo cubo dividido por 6, da un residuo igual al que da su raiz dividida por dicho número 6. Efectivamente, todo número que no es multiple de 6, puede representarse por una de las siguientes espresiones—$x+1$, $x+2$, $x+3$, $x+4$, $x+5$, siendo x igual a cero ó un multiplo de 6.

Si x es igual 0, hallaremos que los cubos de 1, 2, 3, 4, 5 ó 1, 8, 27, 64, 125, divididos por 6, dan por residuo 1, 2, 3, 4, 5.

Si x es un multiplo de 6, tendremos $(x+1)^3=x^3+3x^2+3x+1$, expresion que dividida por 6 dá por residuo 1.

$(x+2)^3=x^3+6x^2+12x+8$, expresion que dividida por 6, nos dá un residuo de 2. Asi mismo se encuentra que $\left.\begin{array}{l}(x+3)^3\\(x+4)^3\\(x+5)^3\end{array}\right\}$ divididos por 6 dan por residuo, 3, 4, 5, y queda demonstrado el teorema.

La extraccion de la raiz cubica es una operacion algo dificil y que exige bastantes cálculos ; asi es que algunos tratados elementales de aritmetica aconsejan que para la extraccion de dichas raices se proceda por tanteo, pero el teorema arriba indicado permite reducir este tanteo á limites muy estrechos tratandose de cubos perfectos. Conociendo el número final de este cubo, desde luego sabremos en que número termina su raiz, y restando de este número el residuo que da la division del cubo por 6, podremos saber en que número hader terminar el multiplo de 6, que agregado al residuo, da la raiz.

Supongamos que se pida la raiz cubica del número 493039. Este cubo acaba en 9 ; por consiquiente su raiz debe acabar en 9. El residuo de la division por 6 es uno, 1, y $9-1=8$, es decir que la raiz debe ser un multiplo de 6 que acaba en 8 mas 1. 493039 es visiblemente menor que el cubo de 80 y mayor que el cubo de 70 : entre 70 y 80, no hay mas que un multiplo de 6 que acaba en 8, que es 78 y desde luego digo sin buscar mas que la raiz es $78+1=79$.

Busquemos la raiz de 35,287,552. La raiz debe acabar en 8. Siendo el residuo de la division por 6, 4, la raiz debera ser formada por un multiplo de 6 que acabe en $4+4$. El dicho cubo es mayor que 300 y menor que 350 como se puede ver por una pequeña multiplicacion. Entre estos dos números no hay mas que un multiplo de 6 que acaba en 4. Es el número 324 y sin mas tanteos digo que la raiz es $324+4=328$.

Theorem.

I found in a book, that the cube of a number less its root, is a multiple of 6, that is to say, that $x^3-x=6n$, but it was not demonstrated. I found the demonstration and tried to get some practical use out of this theorem. I found nothing, but in my investigations I found another theorem which I have applied to the extraction of the cube root. The theorem is the following: Every number divided by 6 gives a remainder equal to that which its cube gives, divided by said number 6; or vice versa, every cube divided by 6 gives a remainder equal to that which its cube root gives, divided by said number 6. Therefore, every number which is not a multiple of 6 may be represented by one of the following expressions: $x+1$, $x+2$, $x+3$, $x+4$, $x+5$, x being equal to zero or a multiple of 6. If x is equal to 0 we will find that the cubes of 1, 2, 3, 4, 5, or 1, 8, 27, 64, 125 divided by 6 give for remainders 1, 2, 3, 4, 5.

If x is a multiple of 6, we will have $(x+1)^3=x^3+3x^2+3x+1$, an expression that divided by 6, gives 1 for a remainder.

$(x+2)^3=x^3+6x^2+12x+8$, an expression that divided by 6, gives us 2 for a remainder. In the same way it is found that $\left.\begin{array}{c}(x+3)^3\\(x+4)^3\\(x+5)^3\end{array}\right\}$ divided by 6 gives for remainders 3, 4, 5 (respectively), and the theorem is demonstrated.

The extraction of the cube root is a somewhat difficult operation, and (one) which requires much calculation; thus it is that some elementary treatises of arithmetic advise that for the extraction of said cubic roots (one) should proceed by approximation, but the theorem above indicated, allows (one) to reduce this approximation to very narrow limits when treating of perfect cubes. Knowing the final number of this cube, consequently we will know in what number its root ends, and subtracting from this number the remainder which the cube divided by 6 gives, we will be able to know in what number the multiple of 6 must end which, added to the remainder, gives the root.

Let us suppose that the cube root of 493039 is asked. This cube ends in 9, consequently its root must end in 9. The remainder of the division by 6 is 1, and $9-1=8$, that is to say that the root must be a multiple of 6 which ends in 8, plus 1. 493039 is plainly less than the cube of 80 and more than the cube of 70; between 70 and 80 there is but one multiple of 6 which ends in 8, which is 78, and consequently I say without any more search that the root is $78+1=79$.

Let us try to find the root of 35,287,552. The root must end by 8. Four, 4, being the remainder of the division by 6, the root must be formed of a multiple of 6 which ends in $4+4$. The said cube is greater than 300, and less than 350, as may be seen by a small multiplication. Between these two numbers there is but one multiple of 6 which ends in 4. It is the number 324, and without more search I say that the root is $324+4=328$.

[Note. This theorem and its translation was furnished by Dr. Halsted. Editor.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJAMIN F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

The following "dissection proofs" differ from the preceding ones only in the fact that there is a displacement of one or more of the squares from their usual places.

LXXXIII. Fig. 34.

Let the $\triangle ABC$ be right-angled at C.

$\triangle LHK = \triangle ABC$, and $\triangle LAE = \triangle HBF$.

$\therefore ABHL \asymp ACDE + DFHK$. Q. E. D.

LXXXIV. Fig. 34.

$LMOA \asymp LKCA \asymp ACDE$.

$HMOB \asymp HKCB \asymp HKDF$.

$\therefore ABHL \asymp ACDE + DFHK$. Q. E. D.

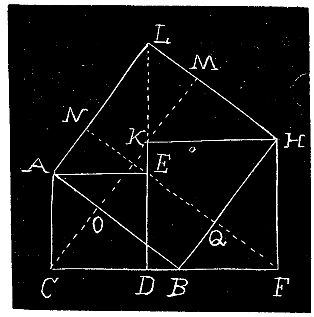


Fig. 34.

LXXXV. Fig. 34.

$ANQB \asymp AEFB \asymp AEDC$.

$LNQH \asymp LEFH \asymp FHKD$.

$\therefore ABHL \asymp ACDE + DFHK$. Q. E. D.

NOTE. In Fig. 34, if ALE is taken as the triangle, we have another type of figure, thus giving several more proofs. See Halsted's *Elements of Geometry*, page 78.

LXXXVI. Fig. 35.

The given triangle is ABC.

$\triangle LNO = \triangle APE'$. $\triangle LMO = \triangle ABC$.

$\triangle AHM = \triangle MDA \asymp PADE + BNK$.

$\therefore AMLB \asymp ADEE' + AHKC$. Q. E. D.

LXXXVII. Fig. 35.

$LRSB \asymp LOCB \asymp ADEE'$.

$AMRS \asymp AMOC \asymp AHKC$.

$\therefore AMLB \asymp ADEE' + AHKC$. Q. E. D.

LXXXVIII. Fig. 35.

$ABTX \asymp ABVD \asymp ADEE'$.

$MLTX \asymp MLVD \asymp AHKC$.

$\therefore AMLB \asymp ADEE' + AHKC$. Q. E. D.

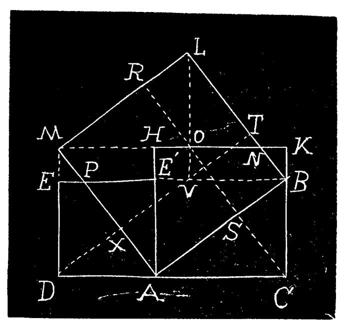


Fig. 35.

NOTE. In Fig. 35, if AMH, MLO, or LBV is taken as the given triangle, we have slightly different types of figures, each yielding various proofs.

LXXXIX. Fig. 36.
$AOC = APE$. $MNB = HFB$.
$MNOL = HKPB$.
$\therefore ABML \simeq ACDE + DFHK$. Q. E. D.

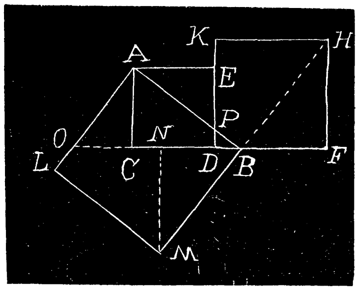


Fig. 36.

XC. Fig. 37.
$PBR = SBK$. $RPLO = ETAD$.
$LNO = ATF$. $NMAC = SBAH$.
$\therefore ABLM \simeq ADEF + AHKC$. Q. E. D.

It will be observed that the above proofs differ chiefly in the relative position of the triangle and the squares. In order to give an idea of still other possible varieties, it may be noted for example, that instead of the arrangement as we have it it Fig. 35, the given triangle may have seven other positions within the square $ACKH$, right angles coinciding. Furthermore, in each case, the square on the hypotenuse may be constructed outwardly from the triangle, or overlapping it. One of these cases, it will be remembered, was considered in November number, 1897.

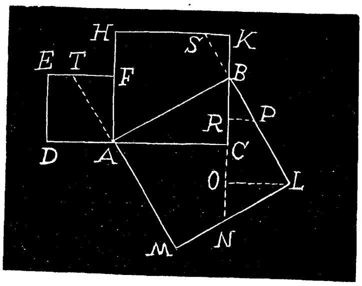


Fig. 37.

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

104. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

If I should buy goods at a price 20% higher than I did buy them, and sell the goods for the same amount that I did sell them, I would gain 25% less than I did gain. What per cent. did I gain? (Solve by Arithmetic).

I. Solution by **W. F. DRADBURY**, Head Master, Cambridge Latin School, Cambridge, Mass.

If I gain 25% less, I get $\frac{6}{5}\times\frac{25}{100}$, or $\frac{150}{500}$ of \$1 less than if I sold at the same per cent. advance. If I sell at same per cent. advance, I should receive $\frac{6}{5}(1+\frac{r}{100})=\frac{6}{5}+\frac{6r}{500}$. What I did receive was $1+\frac{r}{100}$. Subtract, and we have

$\frac{1}{5}+\frac{r}{500}$ loss, or $\frac{1}{5}+\frac{r}{500}=\frac{150}{500}$, $\frac{r}{500}=\frac{50}{500}$, or $r=50$.

Do you call this an Arithmetical solution ? Instead of r, suppose you write "*same*"? Pure Algebra: $1+\frac{r}{100}=\frac{6}{5}\left(1+\frac{r}{100}-\frac{25}{100}\right)$. Then $r=50$.

[NOTE.—According to our definition of an algebraical solution, No. 5, Vol. V., page 139, of the MONTHLY, Professor Bradbury's solution is algebraic. It is immaterial as to what sort of a character is used to represent the quantity sought. It may be a letter, a character of any kind, a word, or several words.

The definition referred to, viz.: Any solution in which the result sought is represented by some character, which character is operated upon until the condition or conditions of the problem are fulfilled which condition or conditions are stated in the form of an equation from which the numerical value of the character is determined, is an algebraic solution,—has received the sanction of some of the best mathematicians in this country. EDITOR F.]

II. Solution by J. OWEN MAHONEY, B. E., Mc., Professor of Mathematics and Science, Carthage Graded and High School, Carthage, Tex.; J. W. YOUNG, Columbus, O.; ELMER SCHUYLER, High Bridge, N. J.; WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.; and the PROPOSER.

Since 100% of the purchasing price is paid for the goods in the first instance, it is easily seen that

the % gain—20%:120::the % gain—25%:100,

or the % gain—20%:5%::120:20

or the % gain—20%:5%::6:1.

∴ The % gain=50.

III. Solution by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

1. Let 100%=actual cost, then
2. 120%=supposed cost.
3. Let *100%*=selling price. [The italics are used for distinction].
4. *100%*—100%=actual gain, and
5. *100%*—120%=supposed gain.
6. $\frac{\mathit{100\%}-100\%}{100\%}-\frac{\mathit{100\%}-120\%}{120\%}=\frac{1}{4}$, or $\frac{\mathit{100\%}-100\%}{100\%}-\frac{\mathit{83\tfrac{1}{3}\%}-100\%}{100\%}=\frac{1}{4}$,

or $\frac{\mathit{16\tfrac{2}{3}\%}}{100\%}=\frac{1}{4}$, whence

7. *16⅔%*=25%; *1%*=1.5%; *100%*=150%, selling price in terms of cost price.
8. ∴ 150%—100%=50%, the gain %.

Also solved by *G. B. M. ZERR* and *B. F. YANNEY*.

105. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A teacher looks at his watch when leaving school at noon. When he comes back he finds that the hour hand and the minute hand have just changed places (that they had when he left the school). What time was it when he left, and what time when he came back to school ? (Solve by Arithmetic.)

I. Solution by JOHN M. ARNOLD, Crompton, R. I.

During the time that the teacher was away from the school, the distance traveled by the minute hand added to the distance traveled by the hour hand,

would make the whole circle of the dial. As the minute hand moves twelve times as fast as the hour hand the spaces passed over would be $\frac{12}{13}$ and $\frac{1}{13}$ of a revolution, respectively.

$\frac{12}{13}$ of a revolution of the minute hand equals $55\frac{5}{13}$ minutes, the time the teacher was away. When the teacher left, the distance between the hands was $\frac{11}{12}$ of the distance of the minute hand from the zero point. Hence $\frac{1}{13}\times\frac{12}{11}=\frac{12}{143}$ $=5\frac{5}{143}$ minutes past twelve, the time when he left. Add the time that he was away, $5\frac{5}{143}+55\frac{5}{13}=60\frac{60}{143}$ or $\frac{60}{143}$ of a minute past one, the time when he returned.

II. Solution by BENJAMIN F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

We first solve the general problem : At what times are the positions of the hands of a watch interchanged ?

It is plain that there are some positions not interchangeable, as, for instance, minute-hand at 6 and hour-hand midway between any two consecutive numbers.

It is evident, also, that in the case of any possible position of interchange, in going from one position to the interchanged one, the two hands must together travel over $60n$ minute spaces, n being an integer.

Then, since the minute-hand travels 12 times as fast as the hour-hand, $\frac{1}{13}$ of the distance traversed by both, or $60n/13$ minute spaces, must be the distance traveled by the hour hand in going from one position of interchange to the other, and $\dfrac{12.60n}{13}$ minute-spaces, the minute-hand's distance. Furthermore, $60n/13$ minute-spaces is the distance in any case of interchange, from hour-hand to minute hand, always reckoned clockwise.

We now have the relative positions of the hands with respect to each other. We next proceed to find the times of these relative and interchanged positions. Suppose the hands at 12. In order that they shall be $60n/13$ minute-spaces apart, the minute-hand must travel over $\dfrac{12.60n}{11.13}$ minute-spaces. This, the number of minute-spaces past 12, gives the time of the first of any two interchanged positions, and $\dfrac{12.60n}{11.13}+\dfrac{12.60n}{13}$ is the time of the corresponding second position.

Now starting, say, with 12 o'clock noon, and substituting for n in succession 1, 2, 3, ...143, which completes a cycle, we shall find, omitting the cases in which the hands are together, 132 different answers to the general problem. We give a few :

$5\frac{5}{143}$ minutes past 12 noon, and $\frac{60}{143}$ of a minute past 1 P. M.
$10\frac{10}{143}$ " " " " " $\frac{120}{143}$ " " " 2 P. M.
....
$\frac{60}{143}$ of a minute past 1 P. M., and $5\frac{5}{143}$ minutes past 12 A. M.
....
$59\frac{83}{143}$ minutes past 10 P. M., and $54\frac{138}{143}$ minutes past 11 P. M.

The first two given are possible answers to the special case in hand.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; B. F. SINE, Principal Capon Bridge Normal School; Capon Bridge, W. Va.; J. D. CRAIG, Frankfort, N. J.; and MARTIN SPINX, Wilmington, O.

Since the two hands had precisely changed positions, they together had passed over all the spaces on the dial-face ; but, as the minute-hand always goes through 60 spaces while the hour-hand goes through 5, both go through 65.

$\therefore$ $65:5=60:4\frac{8}{13}$.

$\therefore$ $4\frac{8}{13}$ spaces is the number of spaces passed over by the hour-hand. This is also the distance the minute-hand was in advance of the hour-hand in the first position.

Since the time he left at noon was after 12 o'clock and since the minute-hand always gains 55 minutes in 60 minutes, to gain $4\frac{8}{13}$ minutes we have $55:60=4\frac{8}{13}:5\frac{5}{143}$.

$\therefore$ The time was $5\frac{5}{143}$ minutes after 12 o'clock. In the second position, the hour-hand was $4\frac{8}{13}$ minutes in advance of the minute-hand. $5\frac{5}{143}-4\frac{8}{13}=\frac{60}{143}$ minutes.

$\therefore$ The time was $\frac{60}{143}$ minutes after 1 o'clock.

$\therefore$ He left at 5 minutes $2\frac{14}{143}$ seconds after 12 o'clock, and returned at $25\frac{35}{143}$ seconds after 1 o'clock.

Also solved by *W. F. BRADBURY, J. W. YOUNG, WALTER H. DRANE, ELMER SCHUYLER*, and *ALOIS F. KOVARIK*.

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithica, N. Y.

Solve by quadratics, $x^2+y=7\ldots\ldots(1)$.
$x+y^2=11\ldots\ldots(2)$.

XI. Solution by W. A. HARSHBARGER, A. M., Professor of Mathematics, Washburn College, Topeka, Kas.

$y^2+x=11\ldots\ldots(1)$, $y+x^2=7\ldots\ldots(2)$.

$(1)-(2)$ $(y^2-x^2)-(y-x)=4\ldots\ldots(3)$.

Put $(y+x)=a$, and $(y-x)=b$.

Then by substituting in (3), $ab-b=4\ldots\ldots(4)$.

$\therefore$ $a^2b^2=16+8b+b^2\ldots\ldots(5)$.

Subtract, $10ab=40+10b$.

$\therefore$ $a^2b^2-10ab=-24-2b+b^2\ldots\ldots(6)$, and

$a^2b^2-10ab+25=1-2b+b^2\ldots\ldots(7)$.

$\therefore$ $ab-5=1-b$, $ab+b=6\ldots\ldots(8)$. $(4)+(8)$, $ab=5$; $(4)-(8)$, $b=1$.

$\therefore$ $a=5$. $\therefore$ $y+x=5$, and $y-x=1$. $\therefore$ $x=2$, and $y=3$.

[NOTE. Professor Harshbarger says the above solution appeared in one of the scientific journals a few years ago, but he has forgotton the name of the author.]

90. Proposed by J. MASCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Solve $$x^2+y^2+m(x+y)=m^2\ldots\ldots(1).$$
$$x^2+y^2+xy=m^2\ldots\ldots(2).$$

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; A. H. BELL, Hillsboro, Ill.; HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; CHARLES C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; and M. A. GRUBER, A. M., Washington, D. C.

Subtracting we have $m(x+y)=xy\ldots\ldots(3)$.

Squaring (3) $m^2(x^2+2xy+y^2)=x^2y^2$.

Multiplying (2) by m^2, $m^2(x^2+xy+y^2)=m^4$.

Whence $x^2y^2-m^2xy=m^4\ldots\ldots(4)$,

Solving as a quadratic, $xy=\dfrac{m^2}{2}(1\pm\sqrt{5})\ldots\ldots(5)$.

Adding (5) and (2), $x^2+2xy+y^2=\dfrac{m^2}{4}(6\pm2\sqrt{5})$.

Whence $x+y=\pm\dfrac{m}{2}(\sqrt{5}\pm1)$.

Similarly, $x-y=\pm\dfrac{m}{2}(\sqrt{-2\mp6\sqrt{5}})$.

Whence $x=\pm\dfrac{m}{4}(\sqrt{5}\pm1+\sqrt{-2\mp6\sqrt{5}})$,

$y=\pm\dfrac{m}{4}(\sqrt{5}\pm1-\sqrt{-2\mp6\sqrt{5}})$,

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1)$-$(2) gives $m(x+y)=xy$. Whence $x+y=xy/m$.

Put $x=vy$. Then $vy+y=vy^2/m$, and $v+1=vy/m$.

Whence $y=m(v+1)/v$, and $x=vy=m(v+1)$.

Substituting the values of x and y in (1) or (2), we have

$$m^2(v+1)^2+\frac{m^2(v+1)^2}{v^2}+\frac{m^2(v+1)^2}{v}=m^2.$$

Dividing by m^2, clearing of fractions, expanding, etc., we obtain

$$v^4+3v^3+4v^2+3v+1=v^2.$$

Adding and subtracting v^3+2v^2+v, we find

$$(v+1)^4-v(v+1)^2=v^2.$$

Completing square and extracting square root, we get

$$2(v+1)^2-v=\pm v\sqrt{5}.$$

Expanding, transposing, and uniting terms, we obtain

$$2v^2+(3\mp\sqrt{5})v=-2.$$

Completing square, extracting square root, etc., we find

$$v=\tfrac{1}{4}[-3\pm\sqrt{5}\pm\sqrt{-2(1\pm3\sqrt{5})}].$$

Whence $v+1=\frac{1}{4}[1\pm\sqrt{5}\pm\sqrt{-2(1\pm3\sqrt{5})}]$,

and $(v+1)/v=\frac{1}{4}[1\pm\sqrt{5}\mp\sqrt{-2(1\pm3\sqrt{5})}]$.

$\therefore$ $x=m(v+1)$ and $y=m[(v+1)/v]$.

III. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa., and ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Let $x=v+z$, and $y=v-z$. $2v^2+2z^2+2mv=m^2$. $3v^2+z^2=m^2$.

Whence $v=\frac{1}{4}(1\pm\sqrt{5})m$, and $z=\pm\frac{1}{4}\sqrt{(-2\mp6\sqrt{5})}m$.

The four values of x and y are

$x=\frac{1}{4}[1+\sqrt{5}+\sqrt{(-2-6\sqrt{5})}]m$; $y=\frac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]m$.

$x=\frac{1}{4}[1+\sqrt{5}-\sqrt{(-2-6\sqrt{5})}]m$; $y=\frac{1}{4}[1+\sqrt{5}+\sqrt{(-2-6\sqrt{5})}]m$.

$x=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]m$; $y=\frac{1}{4}[1-\sqrt{5}-\sqrt{(-2+6\sqrt{5})}]m$.

$x=\frac{1}{4}[1-\sqrt{5}-\sqrt{(-2+6\sqrt{5})}]m$; $y=\frac{1}{4}[1-\sqrt{5}+\sqrt{(-2+6\sqrt{5})}]m$.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $x+y=t$, $xy=r$.

Then (1) and (2) become $t^2-2r+mt=m^2$......(3), $t^2-r=m^2$......(4).

Subtracting (4) from (3) we get $r=mt$......(5), or $t=r/m$......(6).

(5) in (3) gives $t^2-mt=m^2$. $\therefore$ $t=\frac{1}{2}m(1\pm\sqrt{5})$.

(6) in (4) gives $r^2-m^2r=m^4$. $\therefore$ $r=\frac{1}{2}m^2(1\pm\sqrt{5})$.

Since $x+y=t$ and $xy=r$, we have

$x=\frac{1}{2}[t\pm\sqrt{(t^2-4r)}]=\frac{1}{4}m\{1\pm\sqrt{5}\pm\sqrt{[-2(1\pm3\sqrt{5})]}\}$.

$y=\frac{1}{2}[t\mp\sqrt{(t^2-4r)}]=\frac{1}{4}m\{1\pm\sqrt{5}\mp\sqrt{[-2(1\pm3\sqrt{5})]}\}$.

V. Solution by the PROPOSER.

x/m, y/m ; with *eight* roots (four each) [or sixteen *by* $\sqrt{m^2}=\pm m$] ; $x:y=$ ratio $X:Y$ with values $x=X$ times *any*- or *every*-thing; $y=Y$ *any*- or *every*-thing; $\therefore$ $x/X=y/Y=m$ of an infinite degree and roots, *all* values from $\pm(1/\infty^n)$ to 0 to $\pm\infty^\infty$. And that m is a *blind* factor in $x=Xm$; $y=Ym$;—SOLVE CIRODE's curious PROBLEM. (I)−(II)......$m(x+y)=xy$......(III). *Un*-factor $x=Xm$; $y=Ym$......(A). $\therefore$ $X^2m^2+Y^2m^2+m(Xm+Ym)=m^2$......($\mathrm{I}_1$) ; $m(Xm+Ym)=XmYm$......(III_1) *divided* by m^2 are $X^2+Y^2+x+y=1$..... (I_a) ;

$X+Y=XY$......(III$_a$). *Twice* (III$_a$)+(I$_a$) gives $(X+Y)^2-(X+Y)=1$...... (IV). $\therefore X+Y=\frac{1}{2}(1\pm\surd5)$......(V). $\therefore$ By (III$_a$) $XY=\frac{1}{2}(1\pm5)$......(VI) ! $\therefore$ (V)(VI), (by *nature*) the constants of $X^2-\frac{1}{2}(1\pm\surd5)X+\frac{1}{2}(1\pm\surd5)=0$...... (VII) whose *four* roots are by *its origin* (by *turns*) the *eight* roots of *both* X and Y. Multiplying (VII+)(VII−); is $X^4-X^3+2X=0$......(B) *two* real roots each...... $\left\{\begin{array}{l} X=\frac{1}{4}[1-\surd5+\surd(-2+6\surd5)]=Y_1 \\ X_1=\frac{1}{4}[1-\surd5+\surd(-2+6\surd5)]=Y \end{array}\right\}$......(VIII). *Two* imaginery each...... $\left\{\begin{array}{l} X_2=\frac{1}{4}[1-\surd5+\surd(-1-6\surd5)]=Y_a \\ X_a=\frac{1}{4}[1+\surd5-\surd(-2-6\surd5)]=Y_2 \end{array}\right\}$ = 0,809016,994375 $\pm$3.926373,373101,2946$\surd-1$......(IX).

$\therefore \left\{\begin{array}{l} X=-1.153721,375541,766 \text{ to } Y=0.535687......; \\ X_1=0.535687,386791,872 \text{ to } Y_1=-1.153721..... ; \end{array}\right.$

$$\left.\begin{array}{l} X_2=.809016,99......+3.926373,..... \surd-1=Y_a \\ X_a=.809016,99......-3.926373,......\surd-1=Y_2 \end{array}\right\}......(C).$$

$\therefore X+Y=X_1+Y_1=-0.618033,988750-$; $X_2+Y_2=1.618033,988750-=X_a+Y_a$......(D). *Proof* by $XY=X+Y$......(III$_a$) $\left\{\begin{array}{l} X=-1.15372137...... \\ Y=0.53568738...... \end{array}\right\}$

$\left.\begin{array}{l} \text{Product}=-0.6180339,88+=XY \\ \text{Sum} \quad =-0.6180339,89-=X+Y \end{array}\right\}$......(XI). Also (C) and (A).

$\left\{\begin{array}{l} x=Ym=m(-1.1537...... \\ y=Ym=m(\ 0.5356...... \end{array}\right\}$ $\begin{array}{l} \text{Product, } mXmY=xy=in\ x \\ \text{Sum, } \quad mX+mY=x+y \end{array}$

$$\left.\begin{array}{r} y=m_+(x+y)=m^2(-0.61803......) \\ \therefore \text{Multiplying}=m=m(x+y)=m^2(-0.61802\ ....\ ) \end{array}\right\}......(XII),$$

*because (III)..... $xy=m(x+y)=m^2(-0.618033.98......)=m^2XY$......(E). $\therefore$ m (*vanishes*) is only a *blind* factor in x and y. $\therefore$ by (E) ratio $x:y=X:Y$ and and (I) (II) are *fully* solved, process, ratio and roots. Q. E. D. [As, solve (III) for m. $\therefore m=\frac{1}{2}(-1\mp\surd5)(x+y)$...... *i. e.* $1=\frac{1}{2}(-1\mp\surd5)(X+Y)$...... $\therefore m$ and x and y vanish together as *always*].

Corollary. Two quadratics in *three* unknowns, *if* their every term be quadrate (in unknowns) stand for, *and solve*, the two bi-quadratics each in one *true* unknown found by treating two stated unknowns as if multiples (or other functions) of the third, and un-factoring them from it (into two *new* letters). This treatment also determines the third unknown, roots and equation, or their nature and forms when inexpressible.

Corollary 2. *Strictly* $\surd m=\pm m$. $\therefore x^2+y^2\pm(x+y)$......(I$_c$) with (II$_a$) have *eight* roots each, viz.: (VIII, IX) or (C) direct, and *also* with signs reversed.

Scholium. *General* form $x^2+y^2+c(x+y)=a$......(A) ; $x^2+y^2+xy=b$ .....(B) solves (by quadratics) many biquadrates, decomposes surd compounds (other than of 1 with $\surd5$), finds roots, etc.

GEOMETRY.

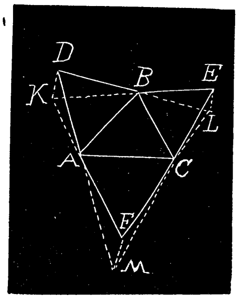
106. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Upon the sides of any triangle ABC let the equilateral triangles ABD, BCE, and CAF be described, and let their exterior sides produced intersect, BE and AF in K, DB and FC in L, and DA and EC in M. Prove DK, EL, FM, parallel.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md., and G. I. HOPKINS, Instructor in Mathematics and Physics, High School, Manchester, N. H.

Angle $KAB=180^\circ-60^\circ-$angle CAB, angle $ABK=180^\circ-60^\circ-$angle ABC.

$\therefore$ Adding, angle $KAB+$angle $ABK=240^\circ-$angle $CAB-$angle ABC. Angle $AKB=180^\circ-$(angle $KAB+$ angle ABK).



$\therefore$ Angle $AKB=$angle $CAB+$angle $ABC-60^\circ$.

Again, angle $BCL=180^\circ-60^\circ-$angle ACB. Angle $ACB=180^\circ-$(angle $CAB+$angle ABC).

$\therefore$ Angle $BCL=$angle $CAB+$angle $ABC-60^\circ$.

$\therefore$ Angle $BCL=$angle AKB.

Angle $MAC=60^\circ+$angle MAF, and angle $KAB=60^\circ+$angle KAD. $\therefore$ Angle $MAC=$angle KAB.

Similarly, angle $MCA=$angle BCL. $\therefore$ Angle $MCA=$angle AKB.

$\therefore$ Triangle AMC is similar to triangle AKB.

$\therefore$ $AC:AK::AM:AB$, or $AF:AK::AM:AD$.

$\therefore$ Triangle AKF is similar to triangle AKD.

$\therefore$ Angle $AMF=$angle ADK.

$\therefore$ KD is parallel to MF. Similarly EL is parallel to MF.

II. Solution by the PROPOSER.

Points K, F, C, B are concyclic. $\because$ $\angle CBE=\angle AFC$.

$\therefore$ $\angle BKA=\angle BCL$. $\angle LBC=\angle KBA$.

$\therefore$ Triangles AKB and CBL are similar, and $KB/AB=LB/CB$, or $KB/EB=LB/EB$.

$\therefore$ KD is parallel to EL.

Similarly $AM/AD=AD/AK$, and therefore FD is parallel to KD.

And therefore EL is parallel to KD is parallel to FM.

III. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

Let ABC be any triangle having equilateral triangles described upon its sides, and their exterior sides produced to intersect BE and AF in K, DB and FC in L, and DA and EC in M. Join FM, DK and EL.

The triangles BCL and ACM are similar, hence $BC:CL::CM:CA$, or $CE:CL::CM:CF$. And since the $\angle ECL=\angle FCM$, the triangles CLE and CFM are similar and equiangular, the angle FMC being equal to the angle LEC.

$\therefore$ ER is parallel to FM......................................(1).

The triangles ABK and AMC are similar, hence $AB : AK :: AM : AC$, or $AD : AK :: AM : AF$.

Since the $\angle DAK = \angle MAF$, the triangles DKA and MFA are similar, and $\angle ADK$ is equal to $\angle AMF$.

$\therefore$ DH is parallel to FM..... ....(2).

$\therefore$ DK, FM and EL are parallel. Q. E. D.

IV. **Solution by CHARLES C. CROSS, Libertytown, Md,**

Draw the figure as indicated in the problem.

Let $\angle BLE = x$, $\angle CEL = y$, $\angle DKB = z$, $\angle ADK = w$, $\angle CEM = v$, and $\angle FMC = w$.

$\angle ECL = 180^\circ - (120^\circ + C) = 60^\circ - C$.

Similarly, $\angle EBL = A - 60^\circ$, and $\angle KAD = 60^\circ - A$.

$\angle BCL = 180^\circ - (60^\circ + C) = 120^\circ - C$.

Similarly, $\angle LBC = 120^\circ - B$, and $\angle BAK = 120^\circ - A$.

Hence $\angle BLC = B + C - 60^\circ$, and $\angle BKA = B + A - 60^\circ$.

$\angle BLE + \angle BLC + \angle CEL + \angle ECL = 180^\circ$; by substitution $B + x + y = 180^\circ$...(1).

$\angle BKA + \angle BKD + \angle KDA + \angle KAD = 180^\circ$; by substitution $B + w + z = 180^\circ$..(2).

From (1) and (2), $x + y = w + z$......................................(3).

If EL and DK are parallel, angle DKB=angle BEL, and angle BLE=angle KDB, or $z = 60^\circ + y$ and $x = 60^\circ + w$. Substituting in (3), $60^\circ + w + y = 60^\circ + w + y$. Hence EL and DK are parallel.

Angle CFM+angle CMF+angle $FCL = 180^\circ$; by substitut'n $v + w - C = 120^\circ$..(4).

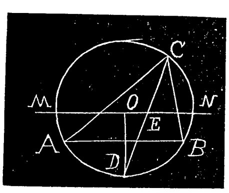
If EL and FM are parallel, then angle MFC=angle ELC, and angle EMC=angle CEL, or $v = x + A + C - 60^\circ$, and $w = y$. Substituting in (4), $A + x + y = 180^\circ$. Since by (1) this relation is true, hence EL and FM are parallel.

107. **Proposed by T. W. PALMER, A. M,, Professor of Mathematics, University of Alabama.**

Construct a triangle, given base, vertical angle and radius of inscribed circle.

I. **Solution by H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.**

Denote the base by AB, the vertex by C, and the incenter by O. The angle AOB equals $90^\circ + \frac{1}{2}C$ and hence one locus for O is the arc of a segment capable of containing this angle. Another locus is a parallel to the base the in-radius away. Hence the incircle can be constructed ; AC and BC are then drawn tangent to it.



II. **Solution by J. SCHEFFER, A. M., Hagersfown, Md.**

Describe on the given base AB a circle the upper segment of which contains the given vertical angle. From the center O of this circle let fall the perpendicular on AB and produce it to D. At a distance from AB equal to the given radius of the inscribed circle draw MN parallel to AB. From D as a center with a radius equal to BD draw an arc cutting MN at E, connect E with D and extend DE until it

cuts the circumference at C, then will ABC be the required triangle. For, since $DE=BD$, 2 angle $EBD=180^\circ-$angle $EDB=180^\circ-A$.

$\therefore$ Angle $EBD=90^\circ-\frac{1}{2}A$, but angle $EBA=$angle $EBD-$angle $ABD=$angle $EBD-\frac{1}{2}C=90^\circ-\frac{1}{2}A-\frac{1}{2}C=\frac{1}{2}B$.

$\therefore$ BE is the bisector of B, and by construction, CD is the bisector of C.

$\therefore$ E is the center of the inscribed circle. Q. E. D.

Also solved by *G. B. M. ZERR, P. S. BERG, COOPER D. SCHMITT, F. H. POWE, F. W. HAMAWALT, ELMER SCHUYLER*, and the *PROPOSER*.

CALCULUS.

81. Proposed by J. OWEN MAHONEY, B. E.. M. Sc., Instructor in Mathematics, Carthage High School, Carthage, Tsxas.

Solve : $y^2(d^2y/dx^2)+a(dy/dx)^2=bx$.

No solution of this problem has been received.

82. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A pole 60 feet high stands vertically in a river 20 feet deep. How many feet above the surface of the water must it break so that the top bending down would touch the bottom and the distance on the surface of water between the points where the parts of the pole enter the water would be a maximum?

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O., and GUY B. COLLIER, Union College, B. S. Course, Schenectady, N. Y.

Let $x=$the number of feet above the surface of the water the pole must break, and $y=$the number of feet between the parts of the pole on the surface of the water, which is to be a maximum.

By similar triangles we find $y=\dfrac{2x}{x+20}\sqrt{300+30x}$.

Simplifying and placing the first derivative equal to zero, we have a biquadratic in x whose roots are : 0, -20, 6.055, and -66.055. By substitution in the second derivative we find that 6.055 is the only one of these roots that renders y a maximum. Therefore $x=6.055$ is the required result.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC represent the pole, BC being the part under water. Let D be the point where it breaks off, so that $DA=DE$. Let $AB=a$, $BC=b$, $BD=x$, $BF=y$; then $DA=DE=a-x$. $CE=\sqrt{[(a-x)^2-(b+x)^2]}=\sqrt{(a+b)}.\sqrt{(a-b-2x)}$ and $CE:y=b+x:x$, whence $y=\sqrt{a+b}.\dfrac{x}{b+x}\sqrt{a-b-2x}$.

$\therefore$ $M=\dfrac{x^2}{(b+x)^2}(a-b-2x)$ is to be a maximum.

By differentiation we obtain after all the necessary and simple transformations the quadratic $x^2+3bx=(a-b)b$, whence $x=\frac{1}{2}[-3b+\sqrt{(5b^2+4ab)}]$.

For the numerical value $a=40$, $b=20$, we get $x=10(\sqrt{13}-3)=6.055$.

III. Solution by W. W. LANDIS, A. M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; O. S. WESTCOTT, A. M., Sc. D., Principal North Division High School, Chicago, Ill.; CHAS. E. MEYERS, Canton, Ohio; NELSON L. RORAY, Bridgetown, N. J.; CHAS. C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; J. H. DRUMMOND, LL. D., Portland, Me.; COOPER D. SCHMITT, A. M. University of Tennessee, Knoxville, Tenn.; and the PROPOSER.

Let x=distance above the water, y=the required maximum.

Then $40-x$ is the hypotenuse and $\sqrt{[(40-x)^2-(20+x)^2]}=2\sqrt{(300-30x)}$ =the base.

$\therefore\ y:x=2\sqrt{(300-30x)}:20+x.$

$\therefore\ y=\dfrac{2x\sqrt{(300-30x)}}{20+x}$=maximum.

Differentiating and reducing we get $(600-90x)(20+x)-2x(300-30x)=0$.

$\therefore\ x^2+60x=400.$

$\therefore\ x=6.0555128$ feet.

MECHANICS.

63. Proposed by A. H. BELL, Hillsboro, Ill.

From a horizontal support at a distance of 10 feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa., and the PROPOSER.

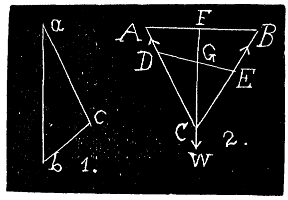
Let $AB=10$, $BC=CD=5$, $AD=3$, $EB=x+y$, $AE=x-y$.

$$\cos AEB=\frac{(x+y)^2+(x-y)^2-100}{2(x+y)(x-y)}=\frac{(x+y-5)^2+(x-y-3)^2-25}{2(x+y-5)(x-y-3)}.$$

$$\therefore\ (139-32x)y^2-(200-8x^2)y-79x^2+800x-1500=0\ \ldots\ldots\ \ldots\ldots\ \ldots(1).$$

$$DF=CN=\frac{100+(x-y)^2-(x+y)^2}{20(x-y)}(x-y-3)=\frac{100+(x+y)^2-(x-y)^2}{20(x+y)}(x+y-5).$$

$$\therefore\ xy^3-xy^2-(x^3-4x^2+100)y+25x=0\ \ldots\ (2).$$



Let $139-32x=c$, $200-8x^2=2a$,
$79x^2-800x+1500=b$, $x^3-4x^2+100=d$.

$$\therefore\ cy^2-2ay-b=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and $xy^3-xy^2-dy+25x=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$

y^2 of (3) in (4) $2axy^2+(bx-2ax-cd)y-bx+25cx=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$

$(5)\times c-(3)\times 2ax,\ y=\dfrac{bcx-2abx-25c^2x}{bcx+4a^2x-2acx-c^2d}$ .... (6).

$(5)\times b+(3)\times(25cx-bx),\ y=\dfrac{b^2x-50acx-bcd}{bcx-2abx-25c^2x}$ .... (7).

Now $b=25c-79a/4$ and $d=x^3+a$ in (6)=(7), etc.

$(316a^2x-558acx)^2+(16a^2x-4c^2x^3-4ac^2-87acx+100c^2x)\times$
$(6241a^2x+316acx^3+316a^2c-400c^2x^3-400ac^2-16600acx+10000c^2x)=0$ .... (8).

Expanding, we find a common factor c, then by substitution and reduction to the simplest form for the application of Horner's Method.

$79x^{10}-1748x^9+12559x^8-2429.5x^7-478451.828125x^6$
$+2827762.5x^5-4080008.59375x^4-27582812.5x^3$
$+161863232.421875x^2-357007812.5x+301890625.=0$ .... (9).

Horner's Method gives $x=7.95690209132$, (3) $y=0.564356664799$.
$x+y=8.521258756118$, $x-y=7.392545426520$.
$CE=3.521258756118$, $DE=4.392545426520$.
$\angle DAB=56^\circ\ 17'\ 54''$, $\angle ABC=46^\circ\ 11'\ 54''$, $\angle AEB=77^\circ\ 30'\ 12''$.
$\angle DCE=59^\circ\ 3'\ 32.5''$, $\angle CDE=43^\circ\ 26'\ 15.5''$.

Tension on $BC=\dfrac{10\cos DAB}{\sin(DAB+ABC)}=5.6863$ pounds.

Tension on $AD=\dfrac{10\cos ABC}{\sin(DAB+ABC)}=7.0896$ pounds.

[NOTE. By a mistake we published the incomplete solution of this problem in our last issue. Soon after receiving that solution, Dr. Zerr wrote us to the effect that a correct and complete solution would shortly follow. Mr. Bell, being ill at the time, was unable to send the complete solution at the time expected, so that by the time the Department was ready for the press, we forgot about the promised complete solution and sent in the incomplete solution for publication. We have not verified the above solution, and our readers must excuse us from that great task. We hold Dr. Zerr and Mr. Bell responsible for any errors contained in it.—ED. F.]

71. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Three men own a sphere of gold the density of which varies as the square of the distance from the center. If two segments be cut off each one inch from the center of the sphere it will be divided into three parts of equal value. Determine the diameter of the sphere.

II. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

The element $dydx$ whose ordinate is y will, when revolved about the axis of x generate an infinitesimal ring whose volume is $2\pi ydydx$, and whose distance from the center is $\sqrt{(x^2+y^2)}$. Therefore for the mass of the minor segment we have

$$2\int_0^1\int_0^{\sqrt{(a^2-x^2)}} 2\pi y(x^2+y^2)dydx=\pi\int_0^1(a^4-x^4)dx=\pi(a^4-\tfrac{1}{5}).$$

Mass of sphere$=\pi\int_0^a(a^4-x^4)dx=\frac{4}{5}\pi a^5$.

$\therefore\ \frac{4}{5}\pi a^5=3\pi(a^4-\frac{1}{5})$, $4a^5=15a^4-3$, which evidently has a root slightly less than 3.75.

In the solution given by Dr. Zerr in the November MONTHLY if the parts be added so as to give the mass of the sphere the result is not homogeneous in a and is therefore evidently wrong. In getting M the upper limit for θ should be $\cos^{-1}(1/r)$ and not $\cos^{-1}(1/a)$. For M_1 we must subtract $2M$ from the mass of the sphere.

75. Proposed by **B. F. FINKEL, A.M.,M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A particle P, is held in a bent tube by two forces directed towards two fixed points, H and S. Show that the equation of the tube is $PS.PH=k^2$, if the forces are μ/PS and μ/PH.

I. Solution by **GEORGE R. DEANE, C. E., B. S., Professor of Mathematics, Missouri School of Mines, Rolla, Mo.**

Put $PS=r_1$, $PH=r_2$. By the principle of virtual work, we have,

$$\frac{\mu}{r_1}\delta r_1+\frac{\mu}{r_2}\delta r_2=0.$$

Let $f(r_1,\ r_2)=0$ be the equation of the curve. Then

$$\frac{\partial f}{\partial r_1}\delta r_1+\frac{\partial f}{\partial r_2}\delta r_2=0.$$

Eliminating δr_1 and δr_2,

$$\frac{\dfrac{\partial f}{\partial r_1}}{\dfrac{\partial f}{\partial r_2}}=\frac{\dfrac{\mu}{r_1}}{\dfrac{\mu}{r_2}}.$$

Whence, $-\dfrac{dr_2}{dr_1}=\dfrac{r_2}{r_1}$, $r_1dr_2+r_2dr_1=0$, $r_1r_2=k^2$.

The general theorem of which this is a particular case, is given in Minchin's *Statics*, Vol. I., page 88.

II. Solution by **WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.**

Let s=any arc of the tube, r, r'=the distances of the particle from the centers of force at any time t, m=the same absolute intensities of the forces, and β =the velocity of projection.

If S, S' be the radial forces attracting the particle, we will have

$$\frac{d^2s}{dt^2}=-\,S\frac{dr}{ds}-S'\frac{dr'}{ds}\ldots\ldots(1).$$

But $S=m/r$, $S'=m/r'$, and (1) becomes

$$\frac{d^2s}{dt^2}=-\,\frac{m}{r}\frac{dr}{ds}-\frac{m}{r'}\frac{dr'}{ds}\ldots\ldots(2).$$

Multiply by $2(ds/dt)$ and integrate ; then

$$\frac{ds^2}{dt^2}=-m\log r^2-m\log r'^2+C\ldots\ldots(3).$$

When $r=a$, $r'=a$, $\dfrac{ds}{dt}=\beta$; $\therefore\ C=\beta^2-m^2\log\dfrac{1}{a^4}$, and (3) is

$$\beta^2=m^2\log\frac{1}{r^2r'^2}+\beta^2-m^2\log\frac{1}{a^4}\ \ldots\ (4),$$

or, $rr'=a^2\ldots\ldots(5)$, a lemniscate.

[Other solutions of this problem will appear in the next issue.]

DIOPHANTINE ANALYSIS.

73. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find integral values for x and y in $\begin{pmatrix}2x^2-y^2=\square\\ 2y^2-x^2=\square\end{pmatrix}$.

I. Solution by the PROPOSER.

$$2x^2-y^2=\square=a^2\ldots\ldots(1),\ 2y^2-x=\square=(2).$$

From (1), $y^2=2x^2-a^2$. Substituting this in (2), we have $3x^2-2a^2=b^2$. Whence $x=\frac{1}{3}\surd[3(2a^2+b^2)]$, and $y=\frac{1}{3}\surd[3(a^2+2b^2)]$.

As far as I know, the only method of rationalizing both radicals is to put $a=b$. Then $x=y=a=b$.

Accordingly, no *different* integral values can be found for x and y.

This problem is the key to Problem 62, "To find four squares in arithmetical progression."

The roots of the four squares would then be, respectively,

$$a,\quad \tfrac{1}{3}\surd[3(2a^2+b^2)],\quad \tfrac{1}{3}\surd[3(a^2+2b^2)],\quad b.$$

The common difference of the squares is $\frac{1}{3}(b^2-a^2)$.

According to the above solution, the roots of the four squares could not *all* be rational integers ; *one* of them, at least, must be a *surd*. It is evident, however, that an infinite number of sets of four squares can be found in which *two* of the roots are rational integers.

Put $a=1$ and $b=2$. Then the roots of the four squares are 1, $\sqrt{2}$, $\sqrt{3}$, 2.

Put $a=1$ and $b=5$. Then the roots are 1, 3, $\sqrt{17}$, 5.

A similar proof was received from *CHARLES C. CROSS.*

II. Solution by A. H. BELL, Hillsboro, Ill.

Take $2y^2-x^2=\square$, or $x^2-2y^2=-\square=-1=-4$, etc.

In $x^2-2y^2=-1\ldots\ldots(3)$, the integral values for x and y are the alternate convergent fractions for the $\sqrt{2}$=to

$$x/y=1/1,\ 7/5,\ 41/29,\ \text{etc}\ldots\ldots(4).$$

For the next, $x^2-2y^2=-4$. $(4)\times\sqrt{4}$,

$$\frac{x}{y}=\frac{1\times2}{1\times2},\quad \frac{7\times2}{5\times2},\quad \frac{41\times2}{29\times2},\quad \text{etc.}$$

Consequently the interchangeable values of x and y must be found in the first fraction and no other.

III. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

$2x^2-y^2=\square\ldots\ldots(1)$, and $2y^2-x^2=\square\ldots\ldots(2)$.

Take $x=my$ and $2m^2-1=\square\ldots\ldots(3)$, and $2-m^2=\square\ldots\ldots(4)$.

Then $m<\sqrt{2}$ and $m>\frac{1}{2}\sqrt{2}$.

It is manifest that both (3) and (4) are rational when $m=1$, which is $<\sqrt{2}$ and $>\frac{1}{2}\sqrt{2}$.

Then in (3) take $m=n+1$, and we have

$$2n^2+4n+1=\square=(\text{say})(qn-1)^2,\ \text{whence}$$

$$n=\frac{2(q+2)}{q^2-2}\ \text{and}\ m=n+1=\frac{(q+1)^2+1}{q^2-2}$$

Substituting this value of m in (4) and reducing by the usual methods, we find $q=0$ and $m=\pm1$.

Hence $x=\pm y$ and the integral values are any equal numbers, positive or negative, or one positive and the other negative.

MISCELLANEOUS.

65. Proposed by J. M. COLAW, A. M., Monterey, Va.

Three circles, radii in ratio 1, 3, 5, are tangent externally and enclose one acre; what are the radii ?

Solution by A. H. BELL, Hillsboro, Ill.; J. SCHEFFER, A. M., Hagerstown, Md.; FREMONT CRANE, Sand Coulee, Mont., and COOPER D. SCHMITT, A. M. University of Tennessee, Knoxville, Tenn.

Let the radii be x, $3x$, and $5x$, respectively.

The sides of the triangle formed by joining the centers are $4x$, $6x$, and $8x$, and let the angles opposite each side, respectively, be A, B, and C; and then are found from the sides of the initial triangle 2, 3, and 4,

$$\cos A=\frac{9+16-4}{24}. \quad \therefore A=28^\circ\ 57'\ 18''\ \text{arc}=.5053601,$$

$$\cos B=\frac{4+16-9}{16}. \quad \therefore B=46^\circ\ 34'\ 3''\ \text{arc}=.8127562,$$

$$\cos C=\frac{4+9-16}{12}=-\tfrac{1}{4}. \quad \therefore C=104^\circ\ 28'\ 39''\ \text{arc}=1.8234763.$$

Now the triangle equals the three seetors=one acre=(10 square chains).

$$\text{Then}\ \frac{25x^2A}{2}+\frac{9x^2B}{2}+\frac{x^2C}{2}+10=24x^2\sin A \ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore x=\sqrt{\frac{20}{48\sin A-(25A+9B+C)}}\ldots\ldots\ldots\ldots\ldots(2).$$

$1-\cos^2 A=\sin^2 A$ and $\sin A=\frac{1}{8}\sqrt{15}=0.4841229$.

$x=3.694$ chains, and the radii are 3.694, 11.082, and 18.47 chains.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

Solved by Elmer Schuyler, with results, 9.985 rods, 29.955 rods, and 49.925 rods; by Alois F. Kovarik with 19.903 rods, 57.709 rods, and 99.515 rods; and by Josiah H. Drummond with approximate results. The methods were all correct but there were some errors of calculation. Later a solution was received from Walter H. Drane, with results, 211.329+ feet, 633.987+ feet, and 1056.645+ feet.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

108. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A man who feels his death approach bequeathes to his young wife one-third of his fortune, and the remaining two-thirds to his son, if such should be born; but one-half of it to the widow and the other half to his daughter, if such should be born. After his death twins are born, a son and a daughter. How should the fortune be divided amongst the three ?

109. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Why do fences and telegraph poles appear to move rapidly in an opposite direction to one traveling in a railway car ? [From Moore's *Grammar School Arithmetic*, page 150.]

⁂ Solutions of these problems should be sent to B. F. Finkel not later than April 10.

ALGEBRA.

96. **Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.**

How many different numbers may be written with the nine digits and zero, using them singly and in groups of from two to ten digits each, and using no figure but once in each group? How many more nnmbers may be written by repeating the digits and zero at pleasure in each group?

97. **Proposed by F. M. SHIELDS, Coopwood, Miss.**

A farmer had 2080 pounds of grain at the depot, and gave a wagoner .75 cents per 100 pounds to haul it, paying him in the *same* grain at the following prices, viz.: 3-10 of the hauling bill was paid in corn at .58 cents per bushel of 56 pounds. 3-5 was paid in wheat at 1.55 cents per pushel of 60 pounds, and the balance of the bill was paid in oats at .36 cents per bushel of 32 pounds, the wagoner not charging for hauling his own grain. The load being delivered, how many bushels of each kind of grain did the wagoner get, and how many bushels of each kind did the farmer have left after paying the wagoner?

*** Solutions of these problems should be sent to J. M. Colaw not later than April 10.

GEOMETRY.

116. **Proposed by P. S. BERG, A. M., Superintendent of Schools, Larimore, N. D.**

Inscribe by rule and compass a regular heptadecagon.

117. **Proposed by GUY B. COLLIER, Schenectady, N. Y.**

If (x', y') and (x'', y'') are the extremities of a pair of conjugate diameters whose eccentric angles are φ' and φ, show that $\varphi'+\varphi=90^\circ$; given $(x', y')=(a\sec\varphi', b\tan\varphi)$. [From Nichols' *Analytical Geometry.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

MECHANICS.

84. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

Two weights P and Q are fastened by a weightless string that is strung over a single movable pulley. P is greater than Q. The weight of the pulley is $2R$. Find the tension of the string, (1) when the friction of the string on the pulley is neglected, (2) when it is considered.

85. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.**

A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $\sqrt{\dfrac{ng}{a}}$, and a particle is projected from its lowest point with such velocity that it can just reach the highest point; prove that the time of describing the first quadrant is $\sqrt{\dfrac{a}{(n+1)g}}\log(\sqrt{n+2}+\sqrt{n+1})$.

86. Prize Problem ; $2.50 for the best solution.

Two spheres of equal size are in motion on a smooth horizontal plane, and, on meeting, their plane of contact coincides with the plane of the meridian. The sphere on the west side is perfectly elastic and weighs $4\frac{1}{2}$ pounds, while previous to the impact it was moving N. 30° E. with a velocity of 15 feet per second. The sphere on the east side is perfectly plastic and weighs $6\frac{1}{2}$ pounds, while previous to the impact it was moving N. 45° W. with a velocity of 10 feet per second. Determine the motions of the spheres after the impact.

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

EDITORIALS.

Our valued contributor, J. M. Bandy, A. M., Ph. D., is now engaged as chief engineer of the Cape Fear and Northern Railroad.

The mathematical text-books formerly published by Leach, Shewell & Sanborn, of Boston, have been purchased by D. C. Heath & Co. Among the most valuable of these text-books are Osborne's *Differential and Integral Calculus*, Nichols' *Analytic Geometry*, and Fine's *Number System of Algebra*.

Through the kindness of a subscriber, who desires that his name should not be mentioned, we are able to offer a prize for the best solution of Problem 86, Mechanics, published in this issue, furnished by a person under the age of twenty-one years. All solutions must be forwarded to B. F. Finkel on or before May 1, 1899. The object of the donor in giving this prize is to create an interest in mathematics among teachers and young people. The first, second, and third solutions in order of neatness and accuracy will be published.

BOOKS AND PERIODICALS.

Plane and Solid Geometry. By James Howard Gore, Ph. D., Professor of Mathematics in Columbian University, Author of Elements of Geodesy, History of Geodesy, Bibliography of Geodesy, etc., etc. 8vo. Cloth, 210 pages. Price, $1.00. New York : Longmans, Green & Co.

The object of this work seems to be to bring the study of Geometry within the minimum time requisite to gain a fair knowledge of it, in order that a proportionate amount of time may be given to other subjects. The author holds that since other sciences, and even language and philosophy, claim disciplinary merit equal to that possessed by mathematics, the time has come when we can afford to hearken to the demands of the utilitarians and give up those refinements in mathematics which have been retained for the mental discipline they bring about, but which are wholly lacking in practical application.

From this point of view, the author has eliminated from this work all propositions that are not of practical value or needed in the demonstrations of such propositions, and thus about half the matter usually found in geometries is omitted. The author makes one very sound statement in his preface, viz., "While symbols and equational statements have the advantage of brevity and convey information to the mind through its most receptive channel,—eye,—still they discourage the use of language, and hence fail to develop by example and precept the employment of accurate and precise forms of expression." While this statement is true, yet the very concise and symbolic statements made in most of our recent texts on geometry offer no serious objection, if the teacher is careful to see that the student does not fall into a careless and slovenly mode of expression.

We do not agree with the author's utilitarian view except in cases where the object of the study is utilitarian in purpose. In college courses, no subject should be studied simply with a view of its practical value in after life. If such were the aim, many subjects, as for example, Latin, Greek, Chemistry, many branches of Mathematics, etc., might be omitted from the college course. But since the object of the course is disciplinary rather than utilitarian, mathematics must and always will receive its proportionate time, and hence many of the modern refinements of geometry should remain that the study of the subject may result in the highest mental development. B. F. F.

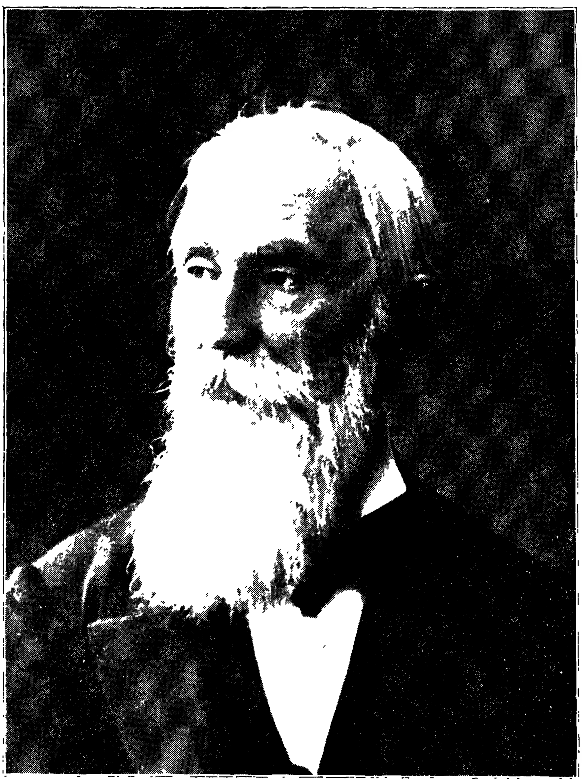
Problèms de Geometrie Élémentaire Groupés D'aprés les Méthodes à Employer pour leur résolutions. Par Ivan Alexandroff, Professeur de Mathematiques au Lycée de Tambov (Russie), Traduit du russie, sur la sixiène èdition, Par D. Aitoff. 8vo. Paper Cover, 156 pages. Paris, France : Librairie Scientifique A. Hermann.

In this work are discussed problems on the construction of geometric figures, methods of geometric loci, method of similitude, method of contrary problems, method of symmetry, method of translation, method of rotation about an axis, method of rotation about a point, method of inversion, and application of algebra to geometry. Under each of these subjects are given demonstrations of type-propositions, that is, propositions setting forth the chief principles employed in the demonstrations of the propositions under the subject considered. For example, under *Problems on the Similitude of Figures*, the following problem comes first: Construct a triangle, having given an angle, the ratio of the sides which contain this angle and the radius of the inscribed circle. This problem is solved and its solution illustrated with a figure. Then comes this problem: Construct a triangle, having given an angle α, the length s of the bisector of this angle and the ratio m/n of the segments determined on the side opposite to α by the altitude from the vertex of this angle. This problem is also solved and the solution illustrated with a figure. Then comes this problem: Having given an angle ABC and a point M on the interior of this angle, find on the side BC a point X equidistant from the side AB and from the point M. Six other problems whose solutions are given follow. These are followed by a large number of exercises. This work in the hands of a good teacher will add interest in his work and profit to his class. B. F. F.

Sur la Résolution de L' Equation du Troisieme Degré. By Dr. Alexander Macfarlane.

In this paper Dr. Macfarlane has applied the hyperbolic calculus to the solution of the cubic equation, and the Method of Cardan and Trigonometric Method are united in a general method applicable to all cases. B. F. F.

Calcul de Généralization. Par G. Oltramare Doyen de la Faculté des Sciences de L' Université de Genéve. Paper Cover, 192 pages. Paris, France : Librairie Scientifique A. Hermann.



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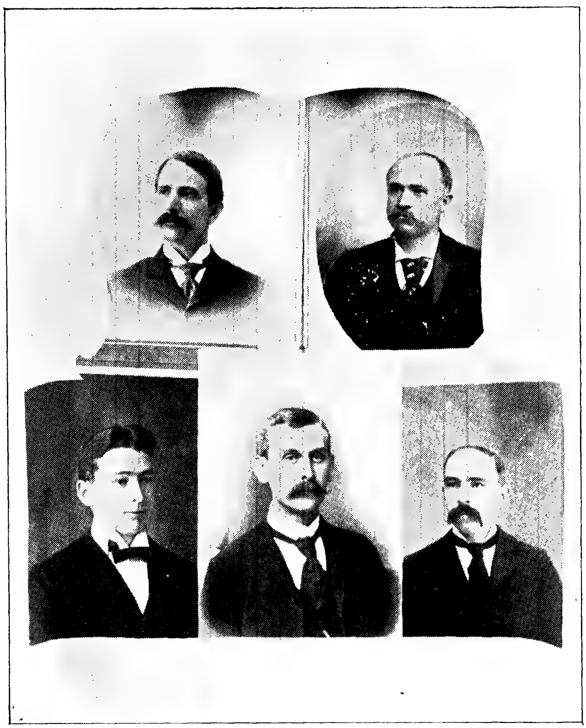

B. F. SINE. J. M. BANDY.
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ON SYMMETRIC FUNCTIONS.

By DR. E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from February Number.]

c. With the first four equations, we have disposed of 16 of the 20 functions. Only four of them needed calculating. They were :

$$b_0^3\Sigma\beta_1^3\beta_2^3,\ \ b_0^2\Sigma\beta_1^2\beta_2^2,\ \ b_0^3\Sigma\beta_1^3,\ \ b_0^2\Sigma\beta_1^2.$$

The remaining 12 were the six functions already mentioned under *a*, and the following six :

$$|\ 0^21\ |=b_0^3\Sigma\beta_1^3\beta_2^3\beta_3^2=b_0^2(\beta_1\beta_2\beta_3)^2b_0\Sigma\beta_1\beta_2,$$

$$|\ 0^22\ |=b_0^3\Sigma\beta_1^3\beta_2^3\beta_3=b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2\beta_2^2,$$

$$|\ 01^2\ |=b_0^3\Sigma\beta_1^3\beta_2^2\beta_3^2=b_0(\beta_1\beta_2\beta_3)^2b_0\Sigma\beta_1,$$

$$|\ 1^22\ |=b_0^3\Sigma\beta_1^2\beta_2^2\beta_3=b_0b_0(\beta_1\beta_2\beta_3)b_0\Sigma\beta_1\beta_2,$$

$$|\ 02^2\ |=b_0^3\Sigma\beta_1^3\beta_1\beta_2=b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2,$$

$$|\ 12^2\ |=b_0^3\Sigma\beta_1^2\beta_2\beta_3=b_0b_0(\beta_1\beta_2\beta_3)b_0\Sigma\beta_1.$$

d. The remaining four functions may be obtained from four of the last six equations. The functions are :

$$|\,012\,| = b_0^3\Sigma\beta_1^3\beta_2^2\beta_3 = b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2\beta_2,$$

$$|\,013\,| = b_0^3\Sigma\beta_1^3\beta_2^2,$$

$$|\,023\,| = b_0^3\Sigma\beta_1^3\beta_2,$$

$$|\,123\,| = b_0^3\Sigma\beta_1^2\beta_2.$$

On account of the decomposition of the first into a known function and one of the other three, only three of these four require actual calculation.

3. CONTINUATION OF THE METHOD.

This method may be continued for higher resultants, what was said under (3) regarding the analysis of the operator δ of Aronhold into three others being capable of the easiest extension to the general case in that in the statements there made one must substitute the numbers 0, 1, 2, $\ldots\ldots n$, and the literal factors $b_0, b_1, \ldots\ldots b_n$ instead of 0, 1, 2, 3, and b_0, b_1, b_2, b_3, respectively, and then these operators must be applied to all the $\frac{(n+1)(n+2)\ldots\ldots(2n-1)}{(n-1)!}$ stroked forms of $(n+1)$ elements to $(n-1)$ dimensions, thus obtaining $\frac{(n+1)(n+2)\ldots.(2n-1)}{(n-1)!}$ identical equations from which the $\frac{(n+1)(n+2)\ldots.2n}{n\,!}$ symmetric functions in connection with the $2n$ which may be already assumed as known, can be found. We have thus at the same time a method for elimination by means of symmetric functions, and a method for computing the values of the symmetric functions, giving the function as a whole. The functions for intermediate forms where m and n are unequal $(m>n)$ are contained in those forms where $n=m$.

II. ISOLATION OF TERMS OF A SYMMETRIC FUNCTION.

A. PRELIMINARY STATEMENTS CONCERNING THE RESULTANT THEORY.

1. THE CONCEPT OF NORMAL AND REDUCIBLE FORMS.

(1). *Definitions.*

In the thesis of the writer entitled "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen," (B. G. Teubner, Leipzig, 1898) the forms

$$P_{m,n} = a_{\kappa_1}a_{\kappa_2}\ldots.a_{\kappa_n}b_{\lambda_1}b_{\lambda_2}\ldots b_{\lambda_m} = (a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu},$$

which occur in the resultant $R_{m,n}$ of

$$f = a_0x^m - a_1x^{m-1} + \ldots.(-1)^m a_m \text{ and } \phi = b_0x^n + b_1x^{n-1} + \ldots. + b_n$$

are divided into such as have the four factors a_0, b_0, a_m, b_n, and those which do

not have all four factors.* The former are called normal forms. It is shown that the latter are capable of being reduced by means of one or more of four kinds of reduction to normal forms of resultants $R_{\mu,\nu}$ of lower order and are called reducible forms, or else are *completely* reducible to a_0 or b_0 (in reality to unity), and are then called completely reducible forms.

(2). *Formulas.*

The normal forms have according to (1) the formula,

$$N_{m,n}=a_0 a_{\kappa_2}\dots a_{\kappa_{n-1}} a_m b_0 b_{\lambda_2}\dots b_{\lambda_{m-1}} b_n=(a_0)^{p_1}(a_{r_2})^{p_2}$$

$$\dots(a_{r_{\mu-1}})^{p_{\mu-1}}(a_m)^{p_n}(b_0)^{q_1}(b_{s_2})^{q_2}\dots(b_{s_{\nu-1}})^{q_{\nu-1}}(b_n)^{q_\nu}.$$

The normal form $N_{\mu,\nu}$ of the resultant $R_{\mu,\nu}$, to which a reducible term

$$P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\dots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\dots(b_{s_\nu})^{q_\nu}$$

of $R_{m,n}$ is reduced, (by means of reductions to be stated in 2) has the formula

$$N_{\mu,\nu}=(a_{r_\lambda-\rho})^{p_\lambda-\alpha}(a_{r_{\lambda+1}-\rho})^{p_{\lambda+1}}\dots(a_{r_\tau-\rho})^{p_\tau-r}(b_{s_\kappa-\sigma})^{q_\kappa-\beta}(b_{s_{\kappa+1}-\sigma})^{q_{\kappa+1}}\dots(b_{s_\pi-\sigma})^{q_\pi-\delta},$$

where $r_\lambda=\rho$, $r_\tau=\mu+\rho$, $s_\kappa=\sigma$, $s_\pi=\nu+\sigma$, and where at least one of the numbers α, β, and at least one of the numbers r, δ is zero. Similarly, but without altering its value, the coefficient

$$C_{m,n}=(r_1)^{p_1}(r_2)^{p_2}\dots(r_\mu)^{p_\mu} \mid (s_1)^{q_1}(s_2)^{q_2}\dots(s_\nu)^{q_\nu}$$

$$\text{of } P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\dots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\dots b_{s_\nu})^{q_\nu}$$

is reduced in form to $C_{\mu,\nu}$, so that $C_{m,n}=C_{\mu,\nu}$, or

$$(r_1)^{p_1}(r_2)^{p_2}\dots(r_\mu)^{p_\mu} \mid (s_1)^{q_1}(s_2)^{q_2}\dots s_\nu)^{q_\nu}$$

$$=(r_\lambda-\rho)^{p_\lambda-\alpha}(r_{\lambda+1}-\rho)^{p_{\lambda+1}}\dots(r_\tau-\rho)^{p_\tau-r} \mid (s_\kappa-\sigma)^{q_\kappa-\beta}(s_{\kappa+1}-\sigma)^{q_{\kappa+1}}\dots(s_\pi-\sigma)^{q_\pi-\delta}$$

The completely reducible forms of $R_{m,n}$ have the general formula :

$$v_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\dots(a_{r_\mu})^{p_\mu}(b_0)^{r_1}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_2}\dots(b_{p_1+p_2+\dots p_\mu})^{m-r_\mu}.$$

2. The Four Kinds of Reduction.

The before-mentioned reductions are attained in the following ways :

(1). *The first reduction.*

By dividing $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\dots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\dots(b_n)^{q_\nu}$, a form of $R_{m,n}$, which contains the factor b_n, but not a_m, by $(b_n)^{m-r_\mu}$, $P_{m,n}$ is reduced to $P_{r_\mu},\ n=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\dots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\dots b_n{}^{q_\nu-m+r_\nu}$, a form of a lower resultant $R_{r_\mu,n}$.

---

*It is a fundamental condition that every form $P_{m,n}$ must contain at least one of the factors a_0, b_0, and at least one of the factors a_m, b_n.

(2). *The second reduction.*

By dividing $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ of $R_{m,n}$, which has the factor a_m but not b_n, by $(a_m)^{n-s_\nu}$, we obtain $P_{m,s_\nu}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu-n+s_\nu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$, a form of the resultant R_{m,s_ν} of lower order than $R_{m,n}$.

(3). *The third reduction.*

Here $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ (with $r_1 \neq 0$) is reduced to $P_{m-r_1,n}=(a_0)^{p_1}(a_{r_2-r_1})^{p_2}\ldots(a_{r_\mu-r_1})^{p_\mu}(b_0)^{q_1-r_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$, a form of the resultant $R_{m-r_1,n}$, by dividing by $(b_0)^{r_1}$ and diminishing the subscripts of the a's by r_1.

(4). *The fourth reduction.*

In this case $P_{m,n}=(a_0)^{p_1}(a_{r_2})^{p_2}\ldots\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots\ldots(b_{s_\nu})^{q_\nu}$, $(s_1 \neq 0)$, is reduced to $P_{m,n-s_1}=(a_0)^{p_1-s_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{q_1}(b_{s_2-s_1})^{q_2}\ldots(b_{s_\nu-s_1})^{q_\nu}$, of the resultant $R_{m,n-s_1}$, by division by $(a_0)^{s_1}$, and diminution of the subscripts of the b's by s_1.

3. THE FOUR KINDS OF DERIVATION.

Conversely we may start with forms of lower resultants and by four kinds of derivation attain to forms of higher resultants.

(1). *The first kind of derivation.*

By multiplying $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ by $b_n{}^q$, we obtain $P_{m+q,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}b_n{}^q$, a form of the resultant $R_{m+q,n}$.

(2). *The second kind of derivation.*

Similarly from $P_{m,n}$ we obtain $P_{m,n+p}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(a_m)^p(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$, of the resultant $R_{m,n+p}$, by multiplication by $a_m{}^p$.

(3). *The third kind of derivation.*

In this case we obtain $P_{m+q,n}=(a_{r_1+q})^{p_1}(a_{r_2+q})^{p_2}\ldots(a_{r_\mu+q})^{p_\mu}(b_0)^q(b_{s_1})^{q_1}\ldots(b_{s_\nu})^{q_\nu}$, of the resultant $R_{m+q,n}$, from $P_{m,n}$ by multiplication by $b_0{}^q$ and increase of the subscripts of the a's by q.

(4). *The fourth kind of derivation.*

Here we multiply $P_{m,n}$ by $a_0{}^p$ and increase the subscripts of the b's by p, and obtain $P_{m,n+p}=(a_0)^p(a_{r_1})^{p_1}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1+p})^{q_1}(b_{s_2+p})^{q_2}\ldots(b_{s_\nu+p})^{q_\nu}$, of the resultant $R_{m,n+p}$.

4. RECURRENCE FORMULA FOR THE NORMAL COEFFICIENTS.

For the calculation of the coefficients of the normal forms we use the formula

$$p_1 \times 0^{p_1}(r_2)^{p_2}\ldots m^{p_\mu} \mid 0^{q_1}(s_2)^{q_2}\ldots n^{q_\nu} = \sum_{\lambda=2}^{\lambda=\nu}(-1)^{s_\lambda+1}(c_\lambda+1)$$

$$\times s_\lambda\, 0^{p_1-1}(r_2)^{p_2}\ldots m^{p_\mu} \mid (s_\lambda)^{-1}0^{q_1+1}(s_2)^{q_2}\ldots n^{q_\nu},$$

where c_λ is the exponent of s_λ in the expression $0^{p_1-1}(r_2)^{p_2}\ldots m^{p_\mu}$. It is a recurrence formula and serves for the calculation of the coefficient of the normal form $(a_0)^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots(b_n)^{q_\nu}$ from earlier calculated coeffi-

cients of simpler forms. In it every coefficient on the right hand is to be reduced to its simplest form according to 1, (2).

5. THE RESULTANT $R_{m,n}$ IN TERMS OF SYMMETRIC FUNCTIONS.

(1). *The expressions.*

The resultant $R_{m,n}$ of f and ϕ may be written in the forms,

$$(-1)^{mn}b_0{}^m f(\beta_1)f(\beta_2)\dots f(\beta_n)=$$

$$(-1)^{mn}b_0{}^n[a_0\beta_1{}^m-a_1\beta_1{}^{m-1}+\dots+(-1)^m a_m][a_0\beta_2{}^m-a_1\beta_1{}^{m-1}+\dots+(-1)^m a_m]$$

$$\dots[a_0\beta_n{}^m-a_1\beta_n{}^{m-1}+\dots+(-1)^m a_m]=a_0{}^n\phi(\alpha_1)\phi(\alpha_2)\dots\phi(\alpha_n)$$

$$=a_0{}^n(b_0\alpha_1{}^n+\dots+b_n)(b_0\alpha_2{}^n+\dots+b_n)\dots(b_0\alpha_m{}^n+\dots+b_n).$$

(2). *The coefficient* $(m-\varkappa_1)(m-\varkappa_2)\dots(m-\varkappa_n)\mid 0^{\lambda_0}1^{\lambda_1}\dots n^{\lambda_n}$).

The coefficient of $a_{m-\kappa_1}a_{m-\kappa_2}\dots a_{m-\kappa_n}$, using the first form of $R_{m,n}$, is $(-1)^{\kappa_1+\kappa_2+\dots+\kappa_n}b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\dots\beta_n{}^{\kappa_n}$. In this again the coefficient $(m-\varkappa_1)$ $(m-\varkappa_2)\dots(m-\varkappa_n)\mid 0^{\lambda_0}1^{\lambda_1}\dots n^{\lambda_n}$ of the product $a_{m-\kappa_1}a_{m-\kappa_2}\dots a_{m-\kappa_n}b_0{}^{\lambda_0}b_1{}^{\lambda_1}$ $\dots b_n{}^{\lambda_n}$ in $R_{m,n}$, is equal numerically to the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}\dots b_n{}^{\lambda_n}$ in $b_0{}^n\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\dots\beta_n{}^{\kappa_n}$, that is,

$$(m-\varkappa_1)(m-\varkappa_2)\dots(m-\varkappa_n)\mid 0^{\lambda_0}1^{\lambda_1}\dots n^{\lambda_n}=(-1)^{\kappa_1+\kappa_2+\dots+\kappa_n}$$

$$\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\dots\dots\dots n^{\lambda_n}\\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\dots\beta_n{}^{\kappa_n}\end{pmatrix},\quad\text{where}\quad\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\dots\dots\dots n^{\lambda_n}\\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\dots\beta_n{}^{\kappa_n}\end{pmatrix}$$

denotes the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}\dots b_n{}^{\lambda_n}$ in $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\dots\beta_n{}^{\kappa_n}$.

Using the second form of the resultant, the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}\dots b_n{}^{\lambda_n}$ is

$$a_0{}^n\Sigma(\alpha_1\alpha_2\dots\alpha_{\lambda_0})^n(\alpha_{\lambda_0+1}\dots\alpha_{\lambda_0+\lambda_1})^{n-1}\dots(\alpha_{\lambda_0+\lambda_1+\dots\lambda_{n-1}}\dots\alpha_{\lambda_0+\dots\lambda_n})^0.$$

This last symmetric function we will express symbolically by

$$a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}(\alpha\lambda_2)^{n-2}\dots(\alpha\lambda_n)^0,$$

where $(\alpha\lambda_r)^{n-r}$ means that λ_r roots (α) have the exponent $n-r$.

In this symmetric function the coefficient

$$(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n) \mid 0^{\lambda_0}1^{\lambda_1}\cdots\cdot n^{\lambda_n}$$

of the product

$$a_{m-\kappa_1}a_{m-\kappa_2}\ldots.a_{m-\kappa_n}b_0^{\lambda_0}b_1^{\lambda_1}\ldots.b_n^{\lambda_n}$$

in $R_{m,n}$ is equal numerically to the coefficient of $a_{m-\kappa_1}a_{m-\kappa_2}\ldots.a_{m-\kappa_n}$ in $a_0^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots.(\alpha\lambda_n)^0$, that is

$$(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n) \mid 0^{\lambda_0}1^{\lambda_1}\ldots.n^{\lambda_n}=(-1)^{mn+\kappa_1+\kappa_2+\ldots\kappa_n}$$

$$\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots & (m-\varkappa_n)\\ a_0^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots. & (\alpha\lambda_n)^0\end{pmatrix}.$$

(3). *Relation between coefficients of terms in symmetric functions.*

By (2) we have two expressions for the coefficient of a term of $R_{m,n}$. By equating these expressions, we obtain a relation between the coefficients of terms of two symmetric functions, namely :

$$\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n}\\ b_0^m\beta_1^{\kappa_1}\beta_2^{\kappa_2}\ldots\ldots\beta_n^{\kappa_n}\end{pmatrix}=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n)\\ a_0^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots.(\alpha\lambda_n)^0\end{pmatrix},$$

which is seen to be true whether f is written with alternating sign or not.

With this paragraph the statements concerning the resultant theory, so far as they relate to this paper are finished. It is now proposed to develop a theory for symmetric functious similar to that sketched for the resultant, and to point out the correspondences between them. Moreover, the theory is developed independently of the results for the resultant, with the exception of the last formula, which will be used.

[To be continued.]

THE COLLECTED MATHEMATICAL PAPERS OF ARTHUR CAYLEY.

By GEORGE BRUCE HALSTED, University of Texas, Austin, Texas.

[4to. 13 Vols., each $6.25. Supplementary Vol. containing Titles of Papers and Index, $2.50. Macmillan.]

This republication by the Cambridge University Press of Cayley's papers in collected form is the most fitting monument of his splendid fame. He must ever rank as one of the greatest mathematicians of all time. Cayley exceedingly appreciated this action of the Syndics of the Press, and seven of the large quarto volumes appeared under his own editorship. As to what these 13 volumes contain, it seems vain to attempt even a summary. They cover the whole range of pure mathematics, algebra, analysis, mathematical astronomy, dynamics, and in particular groups, quadratic forms, quantics, etc., etc.

Though abreast of Sylvester as an analyst, he was, what Sylvester was not, also a geometer. Again and again we find the pure geometric methods of Poncelet and Chasles, though perhaps not full assimilation of that greater one than they, who has now absorbed them—Von Staudt. Cayley not only made additions to every important subject of pure mathematics, but whole new subjects, now of the most importance, owe their existence to him. It is said that he is actually now the author most frequently quoted in the living world of mathematicians.

His name is perhaps most closely linked with the word *invariant*, due to his great brother-in-arms, Sylvester. Boole in 1841 had shown the invariance of all discriminants and given a method of deducing some other such functions. This paper of Boole's suggested to Cayley the more general question, to find "all the derivations of any number of functions which have the property of preserving their form unaltered after any linear transformation of the variables." His first results, relating to what we now call invariants, he published in 1845. A second set of results, relating to what Sylvester called covariants, he published in 1846. Not until four or five years later did Sylvester take up this matter, but then came such a burst of genius that after his series of publications in 1851-54 the giant theory of Invariants and Covariants was in the world completely equipped.

The check came when Cayley, in his second Memoir on Quantics, came to the erroneous conclusion that the number of the asyzygetic invariants of binary quantics beyond the sixth order was infinite, "thereby," as Sylvester says, "arresting for many years the progress of the triumphal car which he had played a principal part in setting in motion." The passages supposed to prove this are marked "*incorrect*" in the Collected Mathematical Papers. But this error was not corrected until 1869, [Crelle, vol. 69., pp. 323-354] by Gordan in his Memoir [dated 8th June, 1868], 'Beweis dass jede Covariante und Invariante einer

binaeren Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.'

Cayley at once returned to the question, found the source of his mistake, the unspected and so neglected interdependence of certain syzygies, and devoted his Ninth Memoir on Quantics (7th April, 1870) to the correction of his error and a further development of the theory in the light of Gordan's results.

The whole of this primal theory of invariants may now be regarded as a natural and elegant application of Lie's theory of continuous groups. The differential parameters, which in the ordinary theory of binary forms enable us to calculate new invariants from known ones, appear in a simple way as differential invariants of certain linear groups. The Lie theory may be illustrated by a simple example:

Consider the binary quadratic form

$$f\equiv a_0x^2+2a_1xy+a_2y^2.$$

Applying to f the linear transformation

$$x=\alpha x'+\beta y', \quad y=\gamma x'+\delta y', \tag{1}$$

we obtain the quadratic form

$$f'\equiv a'_0x'^2+2a'_1x'y'+a'_2y'^2,$$

where the coefficients are readily found to be

$$\begin{cases} a'_0=\alpha^2a_0+2\alpha\gamma a_1+\gamma^2a_2, \\ a'_1=\alpha\beta a_0+(\alpha\delta+\beta\gamma)a_1+\gamma\delta a_2, \\ a'_2=\beta^2a_0+2\beta\delta a_1+\delta^2a_2. \end{cases} \tag{2}$$

We may easily verify the following identity:

$$a'_0a'_2-a'^2_1=(\alpha\delta-\beta\gamma)^2(a_0a_2a_1^2).$$

Hence $a_0a_2-a_1^2$ is an invariant of the form f. In the group theory it is an invariant of the group of linear homogeneous transformations (2) on the three parameters a_0, a_1, a_2.

The only covariant of f is known to be f itself. In the Lie theory it appears as the *invariant* of a linear homogeneous group on five variables, x, y, a_0, a_1, a_2, the transformations being defined by the equations (2) together with (1) when inverted.

In general, the invariants of a binary form of degree n are defined by a linear homogeneous group on its $n+1$ coefficients, its covariants by a group on $n+3$ variables.

As in all problems in continuous groups, the detailed developments are greatly simplified by employing the infinitesimal transformations of the groups

concerned. It is readily proven by the group theory that all invariants and covariants are expressible in terms of a finite number of them. This result is, however, not equivalent to the algebraic result that all rational integral invariants (including covariants) are expressible rationally and integrally in terms of a finite number of such invariants.

Twenty years ago, in my "Bibliography of Hyper-Space and Non-Euclidean Geometry," (American Journal of Mathematics, Vol. I., Nos. 2 and 3, 1878) I cited seven of Cayley's papers written before 1873 :

I. Chapters in the Analytical Geometry of (n) Dimensions. Camb. Math. Jour., Vol, IV., 1845, pp. 119-127.

II. Sixth Memoir on Quantics. Phil. Trans., vol. 149, pp. 61-90, (1859).

III. Note on Lobatchewsky's Imaginary Geometry. Phil. Mag. XXIX,. pp. 231-233, (1865).

IV. On the rational transformation between two spaces. Lond. Math. Soc Proc. III., pp. 127-180, (1869-71).

V. A Memoir on Abstract Geometry. Phil. Trans. CLX., pp. 51-63, (1870).

VI. On the superlines of a quadric surface in five dimentional space. Quarterly Journ., Vol. XII., pp. 176-180, (1871-72).

VII. On the Non-Euclidean Geometry. Clebsch. Math. Ann. V., pp. 630-634, (1872).

Four of these pertain to Hyper-Space, and in that Bibliography I quoted Cayley as to its geometry as follows:

"The science presents itself in two ways,—as a legitimate extention of the ordinary *two-* and *three-*dimensional geometries; and as a need in these geometries and in analysis generally. In fact, whenever we are concerned with quantities connected together in any manner, and which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as coördinates of a point in a plane or in space : for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation : but this can only be obtained by means of the notion of a space of the proper dimensionality : and to use such a representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions : the conditions imply relations between the coefficients in the equation of the curve ; and for the better understanding of these relations it was expedient to consider the coefficients as the coördinates of of a point in a space of the proper dimensionality."

For a dozen years after it was written, the Sixth Memoir on Quantics would not have been enumerated in a Bibliography of non-Euclidean geometry, for its author did not see that it gave a generalization which was identifiable with that initiated by Bolyai and Lobachévski, though afterwards, in his Address to the British Association, 1883, he attributes the fundamental idea involved to

Riemann, whose paper was written in 1854. Says Cayley: "In regarding the physical space of our experience as possibly non-Euclidean, Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four dimensional space."

What the Sixth Memoir was meant to do was to base a generalized theory of metrical geometry on a generalized definition of distance. As Cayley himself says: " . . . the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connection with another figure, viz., the conic termed the absolute."

The fundamental idea that a metrical property could be looked at as a projective property of an extended system had occurred in the French school of geometers. Thus Laguerre (1853) so expresses an angle. Cayley generalized this French idea, expressing all metrical properties as projective relations to a fundamental configuration.

We may illustrate by tracing how Cayley arrives at his projective definition of distance. Two projective primal figures of the same kind of elements and both on the same bearer are called conjective. When in two conjective primal figures one particular element has the same mate to whichever figure it be regarded as belonging, then every element has this property. Two conjective figures such that the elements are mutually paired (coupled) form an involution. If two figures forming an involution have self correlated elements, these are called the double elements of the involution. An involution has at most two double elements; for were three self-correlated, all would be self-correlated. If an involution has two double elements these separate harmonically any two coupled elements. An involution is completely determined by two couples.

From all this it follows that two point-pairs A, B and A_1, B_1 define an involution whose double points D, D_1 are determined as that point-pair which is harmonically related to the two given point-pairs. Let the pair A, B be fixed and called the Absolute. Two new points A_1, B_1 are said (by definition) to be equidistant from a double point D defined as above. D is said to be the 'center' of the pair A_1, B_1. Inversely, if A_1 and D be given, B_1 is uniquely determined. Thus starting from two points P and P_1, we determine P_2 such that P_1 is the center of P and P_2, then determine P_3 so that P_2 is the center of P_1 and P_3, etc.; also in the opposite direction, we determine an analogous series of points P_{-1}, P_{-2}, ... We have therefore a series of points

$$\ldots\ldots, P_{-2}, P_{-1}, P, P_1, P_2, P_3, \ldots\ldots$$

at 'equal intervals of distance.' Taking the points P, P_1 to be indefinitely near to each other, the entire line will be divided into a series of equal infinitesimal elements.

In determining an analytic expression for the distance of the two points, Cayley introduced the inverse cosine of a certain function of the coördinates, but

in the Note which he added in the Collected Papers he recognizes the improvement gained by adopting Klein's assumed definition for the distance of any two points P, Q: dist. $(PQ)=c \log\frac{AP.BQ}{AQ.BP}$, where A, B are the two fixed points giving the Absolute.

This definition preserves the fundamental relation

$$\text{dist. }(PQ)+\text{dist. }(QR)=\text{dist. }(PR).$$

In this note (Col. Math. Papers, Vol. 2, p. 604) Cayley discusses the question whether the new definitions of distance depend upon that of distance in the ordinary sense, since it is obviously unsatisfactory to use one conception of distance in defining a more general conception of distance. His earlier view was to regard coördinates "not as distances or ratios of distances, but as an assumed fundamental notion, not requiring or admitting of explanation." Later he regarded them as "mere numerical values, attached arbitrarily to the point, in such wise that for any given point the ratio $x:y$ has a determinate numerical value," and inversely.

But in 1871 Klein had explicitly recognized this difficulty and indicated its solution. He says: "The cross ratios (the sole fixed elements of projective geometry) naturally must not here be defined, as ordinarily happens, as ratios of sects, since this would assume the knowledge of a measurement. In von Staudt's Beitraegen Zur Geometrie der Lage (§ 27, n. 393), however, the necessary materials are given for defining a cross ratio as a pure number. Then from cross ratios we may pass to homogeneous point—and plane—coördinates, which indeed are nothing else than the relative values of certain cross ratios, as von Staudt has likewise shown (Beitrage, § 29, n. 411)."

This solution was not satisfactory to Cayley, who did not think the difficulty removed by the observations of von Staudt that the cross ratio (A, B, P, Q) has "independently of any notion of distance the fundamental properties of a numerical magnitude, viz., any two such ratios have a sum and also a product, such sum and product being each of them a like ratio of four points determinable by purely descriptive construction."

Consider, for example, the product of the ratios (A, B, P, Q) and (A', B', P', Q'). We can construct a point R such that $(A', B', P', Q')=(A, B, Q, R)$. The product of (A, B, P, Q) and (A, B, Q, R) is said to be (A, B, P, R). This last definition of a product of two cross ratios, Cayley remarks, is in effect equivalent to the assumption of the relation

$$\text{dist. }(PQ)+\text{dist. }(QR)=\text{dist. }(PR).$$

The original importance of this memoir to Cayley lay entirely in its exhibiting metric as a branch of descriptive geometry. That this generalization of distance gave pangeometry was first pointed out by Klein in 1871. Klein showed that if Cayley's Absolute be real, we get Bolyai's system; if it be imagi-

nary, we get either spheric or a new system called by Klein single elliptic; if the Absolute be an imrginary point pair, we get parabolic geometry, and if, in particular, the point pair be the circular points, we get ordinary Euclid.

It is maintained by B. A. W. Russell in his powerful Essay on the Foundations of Geometry (Cambridge, 1897) "that the reduction of metrical to projective properties, even when, as in hyperbolic geometry, the Absolute is real, is only apparent, and has merely a technical validity."

Cayley first gave evidence of acquaintance with non-Euclidean geometry in 1865 in the paper in the Philosophical Magazine above mentioned. Though this is six years after the Sixth Memoir, and though another six was to elapse before the two were connected, yet this is, I think, the very first appreciation of Lobachévski in any mathematical journal. Baltzer has received deserved honor for in 1866 calling the attention of Hoüel to Lobachévski's 'Geometrischĕ Untersuchungen,' and from the spring thus opened actually flowed the flood of ever broadening non-Euclidean research. But whether or not Cayley's path to these gold-fields was ever followed by anyone else, still he had therein marked out a claim for himself a whole year before the others.

In 1872 after the connection with the Sixth Memoir had been set up, Cayley takes up the matter in his paper in the Mathematische Annalen 'On the Non-Euclidean Geometry,' which begins as follows: "The theory of the Non-Euclidian Geometry as developed in Dr. Klein's paper, 'Ueber die Nicht-Euclidische Geometrie,' may be illustrated by showing how in such a system we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the 'hyperbolic' case of plane geometry; viz., the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to the points *within* the circle. I use the simple letters, a, A, . . to denote (linear or angular) distances measured in the ordinary manner; and the same letters with a superscript stroke $\bar{a}$, $\bar{A}$, . . to denote the same distances measured according to the theory. The radius of the absolute is for convenience taken to $=1$, the distance of any point from the center can therefore be represented as the sine of an angle.

The distance $\overline{BC}$, or say $\bar{a}$, of any two points B, C is by definition as follows:

$$\bar{a}=\tfrac{1}{2}\log\frac{BI.CJ}{BJ.CI}$$

(where I, J are the intersections of the line BC with the circle)."

As for the trigonometry, "the formulae are in fact similar to those of spherical trigonometry with only $\cosh\bar{a}$, $\sinh\bar{a}$, etc., instead of $\cos a$, $\sin a$, etc."

Cayley returned again to this matter in his celebrated Presidential Address to the British Association (1883), saying there: "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. There is a like system of non-Euclidean solid

geometry." "But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows ?"

The very next year this ever interesting subject recurs in the paper (May 27, 1884) "On the Non-Euclidean Plane Geometry." "Thus the geometry of the pseudo-sphere, using the expression straight line to denote a geodesic of the surface, is the Lobatschewskian geometry ; or rather I would say this in regard to the metrical geometry, or trigonometry, of the surface ; for in regard to the descriptive geometry, the statement requires some qualification . . . this is not identical with the Lobatschewskian geometry, but corresponds to it in a manner such as that in which the geometry of the surface of the circular cylinder corresponds to that of the plane.

I would remark that this realization of the Lobatschewskian geometry sustains the opinion that Euclid's twelfth axiom is undemonstrable."

But here this necessarily brief notice must abruptly stop.

Cayley in addition to his wondrous originality was assuredly the most learned and erudite of mathematicians. Of him in his science it might be said, he knew everything, and he was the very last man who ever will know everything. His was a very gentle, sweet character. Sylvester told me he never saw him angry but once, and that was (both were practicing law !) when a messenger broke in on one of their interviews with a mass of legal documents, new business for Cayley. In an excess of disgust, Cayley dashed the documents upon the floor.

Austin, Texas, February, 1899.

NOTE ON SPHERICAL GEOMETRY.

By G. B. M. ZERR, A. M., Ph. D., Chester, Pa.

DEFINITION. Two arcs of great circles drawn from the vertex of a spherical triangle making equal angles with the spherical bisector of the angle at that vertex are called *isogonal conjugate arcs.* If three arcs drawn through the vertices of a spherical triangle are concurrent, their *isogonal conjugates* with respect to the angles at these vertices are also concurrent.

Let the arcs AM_a, BM_b, CM_c be concurrent at M. To prove that their isogonal conjugates AK_a, BK_b, CK_c are concurrent.

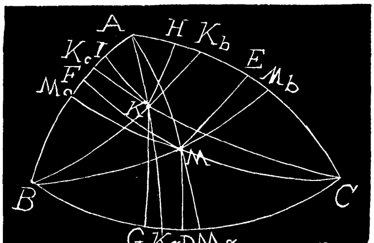


Fig. 1.

Fig. 1. Let $BM_a=a_1$, $CM_a=a_2$, $CM_b=b_1$, $AM_b=b_2$, $AM_c=c_1$, $BM_c=c_2$, $BK_a=a_3$, $CK_a=a_4$, $CK_b=b_3$, $AK_b=b_4$, $AK_c=c_3$, $BK_c=c_4$.

$\angle CMM_b=x$, $\angle CMM_a=y$, $\angle BMM_a=z$, $\angle CAM_a=\angle BAK_a=\theta$, $\angle BAM_a=$

$\angle CAK_a = A-\theta$, $\angle CBM_b = \angle ABK_b = \varphi$, $\angle ABM_b = \angle CBK_b = B-\varphi$, $\angle BCM_c = \angle ACK_c = \psi$, $\angle ACM_c = \angle BCK_c = C-\psi$.

Now we have

$$\sin a_1 : \sin BM = \sin z : \sin M_a$$
$$\sin a_2 : \sin CM = \sin y : \sin M_a.$$

$$\therefore \frac{\sin a_1}{\sin a_2} = \frac{\sin BM \sin z}{\sin CM \sin y} \quad \ldots\ldots (1).$$

Similarly, $$\frac{\sin b_1}{\sin b_2} = \frac{\sin CM \sin x}{\sin AM \sin z} \quad \ldots\ldots (2).$$

$$\frac{\sin c_1}{\sin c_2} = \frac{\sin AM \sin y}{\sin BM \sin x} \quad \ldots\ldots (3).$$

Multiplying (1), (2), (3) together we get as the condition of concurrence, he following :

$$\frac{\sin a_1 \sin b_1 \sin c_1}{\sin a_2 \sin b_2 \sin c_2} = 1 \quad \ldots\ldots (4).$$

Now $$\sin a_3 : \sin AK_a = \sin\theta : \sin B$$
$$\sin a_4 : \sin AK_a = \sin(A-\theta) : \sin C.$$

$$\therefore \frac{\sin a_3}{\sin a_4} = \frac{\sin\theta \sin C}{\sin(A-\theta)\sin B}.$$

But $$\sin a_1 : \sin AM_a = \sin(A-\theta) : \sin B$$
$$\sin a_2 : \sin AM_a = \sin\theta : \sin C.$$

$$\therefore \frac{\sin\theta}{\sin(A-\theta)} = \frac{\sin a_2 \sin C}{\sin a_1 \sin B}. \qquad \therefore \frac{\sin a_3}{\sin a_4} = \frac{\sin a_2 \sin^2 C}{\sin a_1 \sin^2 B}.$$

Similarly, $$\frac{\sin b_3}{\sin b_4} = \frac{\sin b_2 \sin^2 A}{\sin b_1 \sin^2 C}, \qquad \frac{\sin c_3}{\sin c_4} = \frac{\sin c_2 \sin^2 B}{\sin c_1 \sin^2 A}.$$

$$\therefore \frac{\sin a_3 \sin b_3 \sin c_3}{\sin a_4 \sin b_4 \sin c_4} = \frac{\sin a_2 \sin b_2 \sin c_2}{\sin a_1 \sin b_1 \sin c_1} = 1 \quad \ldots\ldots (5).$$

$\therefore$ AK_a, BK_b, CK_c are concurrent at K.

The two points M, K are called *spherical isogonal conjugate points* with respect to the triangle.

Let MD, ME, MF, KG, KH. KI be the spherical distances (perpendicular arcs) of M, K from the sides a, b, c of the triangle.

Then if $\sin MD : \sin ME : \sin MF = \beta : \gamma : \delta$, it can be demonstrated that $\sin KG : \sin KH : \sin KI = 1/\beta : 1/\gamma : 1/\delta$, as follows :

$$\sin MD = \sin CM \sin\psi,$$
$$\sin ME = \sin CM \sin(C-\psi).$$

$$\therefore \frac{\sin MD}{\sin ME} = \frac{\sin\psi}{\sin(C-\psi)} = \beta/\gamma. \quad \therefore \sin MD : \sin ME = \beta : \gamma.$$

$$\sin KG = \sin CK \sin(C-\psi),$$
$$\sin KH = \sin CK \sin\psi.$$

$$\therefore \frac{\sin KG}{\sin KH} = \frac{\sin(C-\psi)}{\sin\psi} = \gamma/\beta. \quad \therefore \sin KG : \sin KH = \gamma : \beta = 1/\beta : 1/\gamma.$$

In the same way it can be shown that $\sin KH : \sin KI = 1/\gamma : 1/\delta$.

$\therefore \sin KG : \sin KH : \sin KI = 1/\beta : 1/\gamma : 1/\delta$.

Definition. If two arcs of great circles are drawn from the vertex of a spherical triangle cutting the base equally distant from the mid-point, the two arcs thus drawn are called *isotomic conjugate arcs*.

If any three arcs drawn from the vertices A, B, C of a spherical triangle to the opposite sides are concurrent, their *isotomic* conjugates are also concurrent.

This follows at once from (5) since $a_2 = a_3$, $a_1 = a_4$, $b_2 = b_3$, $b_1 = b_4$, $c_2 = c_3$, $c_1 = c_4$.

The two points thus determined are called *spherical isotomic conjugate points*.

Fig. 2. Let RD, RE, RF, PG, PH, PI be the spherical distances of R, P from the sides a, b, c of the triangle.

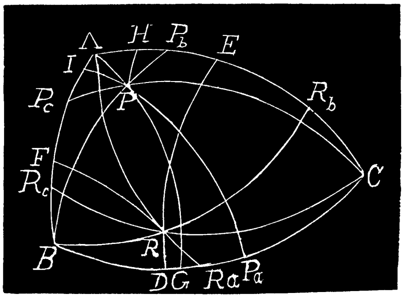


Fig. 2.

Then if $\sin RD : \sin RE : \sin RF = l : m : r$ it can be demonstrated that $\sin PG : \sin PH : \sin PI = \dfrac{1}{l\sin^2 a} : \dfrac{1}{m\sin^2 b} : \dfrac{1}{n\sin^2 c}$ as follows:

Let $BR_a = CP_a = a_1$, $BP_a = CR_a = a_2$,
$CR_b = AP_b = b_1$, $CP_b = AR_b = b_2$,
$BR_c = AP_c = c_1$, $BP_c = AR_c = c_2$.

Then $\sin RD = \sin RC \sin RCD$,
$\sin RE = \sin RC \sin RCE$.

$$\therefore \frac{\sin RD}{\sin RE} = \frac{\sin RCD}{\sin RCE}.$$

But
$$\sin RCD : \sin R_c = \sin c_1 : \sin a$$
$$\sin RCE : \sin R_c = \sin c_2 : \sin b.$$

$$\therefore \frac{\sin RCD}{\sin RCE} = \frac{\sin b \sin c_1}{\sin a \sin c_2}. \quad \therefore \frac{\sin RD}{\sin RE} = \frac{\sin b \sin c_1}{\sin a \sin c_2} = l/m.$$

$$\sin PG = \sin PC \sin PCG$$
$$\sin PH = \sin PC \sin PCH.$$

$$\therefore \frac{\sin PG}{\sin PH} = \frac{\sin PCG}{\sin PCH}.$$

But $\sin PCG : \sin P_c = \sin c_2 : \sin a$
$\sin PCH : \sin P_c = \sin c_1 : \sin b.$

$$\therefore \frac{\sin PCG}{\sin PCH} = \frac{\sin b \sin c_2}{\sin a \sin c_1}. \quad \therefore \frac{\sin PG}{\sin PH} = \frac{\sin b \sin c_2}{\sin a \sin c_1} = \frac{\sin^2 b}{\sin^2 a}.m/l.$$

$$\therefore \sin PG : \sin PH = \sin^2 b.m : \sin^2 a.l \text{ or } \sin PG : \sin PH = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b}.$$

In the same way it can be demonstrated that

$$\sin PH : \sin PI = \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}.$$

$$\therefore \sin PG : \sin PH : \sin PI = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}.$$

Examples. If M is the median point, K is the symmedian point.
In this case $\frac{\sin(C-\psi)}{\sin\psi} = \frac{\sin A}{\sin B}$.
$\therefore \sin MD : \sin ME : \sin MF = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C$
$\sin KG : \sin KH : \sin KI = \sin A : \sin B : \sin C.$

If R is the point of concurrence of arcs drawn from the angles to the points of contact of the incircle with the opposite sides ; P, its isotomic conjugate point, is the point of concurrence of arcs drawn from the angles to the points of contact of the ex-circles with the opposite sides.

In this case $c_1 = (s-b)$, $c_2 = (s-a)$.

$$\therefore \frac{\sin RD}{\sin RE} = \frac{\sin b \sin(s-b)}{\sin a \sin(s-a)} = \frac{\cos^2 \frac{1}{2}B}{\cos^2 \frac{1}{2}A} = \frac{\sec^2 \frac{1}{2}A}{\sec^2 \frac{1}{2}B}.$$

$$\therefore \sin RD : \sin RE : \sin RF = \text{sic}^2 \tfrac{1}{2}A : \sec^2 \tfrac{1}{2}B : \sec^2 \tfrac{1}{2}C.$$

$$\therefore \sin PG : \sin PH : \sin PI = \frac{\cos^2 \frac{1}{2}A}{\sin^2 a} : \frac{\cos^2 \frac{1}{2}B}{\sin^2 b} : \frac{\cos^2 \frac{1}{2}C}{\sin^2 c}.$$

If we start with the in-center, whose distances from the sides are $1:1:1$, and take its isotomic conjugate we get a point whose distances from the sides are

$$\frac{1}{\sin^2 a} : \frac{1}{\sin^2 b} : \frac{1}{\sin^2 c}.$$

The isogonal conjugate of this last point is a point whose distances from the sides are $\sin^2 a : \sin^2 b : \sin^2 c$. By repeating this process we get a series of points whose distances from the sides are as $\sin^m a : \sin^m b : \sin^m c$, where m is an even positive or negative integer. For the symmedian point we have $\sin A : \sin B : \sin C = \sin a : \sin b : \sin c$. Starting with this point and alternating as above, we get a series of points whose distances from sides are as $\sin^n a : \sin^n b : \sin^n c$ where n is any odd positive or negative integer.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJAMIN F. YANNEY, A. M., Mount Union College. Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from February Number.]

XCI. Fig. 38 and Fig. 39.

The triangle is ABC. The construction in the two figures is evident. The Hindu Bhaskara, the author of this method, complimented his readers by condensing his proof into the single word, "Behold." We follow his example.

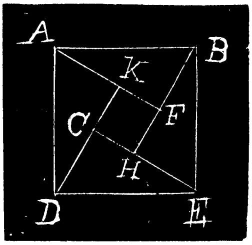


Fig. 38.

NOTE. The above is a conjectured proof of Pythagoras. See pages 50 and 123 of Cajori's "History of Elementary Mathematics."

XCII. Fig. 38.

$$\begin{aligned} AB^2 &= 2AC.BC+CH^2 \\ &= 2AC.BC+(AH-AC)^2 \\ &= 2AC.BC+(BC-AC)^2 \\ &= BC^2+AC^2. \end{aligned}$$

Q. E. E.

XCIII. Fig. 38.

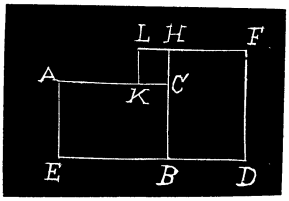


Fig. 39.

Suppose BC is produced to meet AD, as at L.

Then let LBA be the given triangle, right-angled at A.

Now, the the area of the square on AB=the sum of the four triangles ABC, BEK, DEF and ADH, and the square $CHFK$; or, $AB^2=2AC.BC+CH^2$.

Again, $AC=(AL.AB)\div BL$, $BC=AB^2\div BL$, and $CH=AH-AC=BC-AC$.

$$\therefore\ 2AC.BC+CH^2=BC^2+AC^2=\frac{AB^4}{BL^2}+\frac{AL^2.AB^2}{BL^2}.$$

$$\therefore\ BL^2=AB^2+AL^2.$$

Q. E. D.

XCIV. Fig. 40.

$AB^2=CF^2-2AC.BC$

$=(BF+BC)^2-2AC.BC$

$=(AC+BC)^2-2AC.BC$

$=AC^2+BC^2$. Q. E. D.

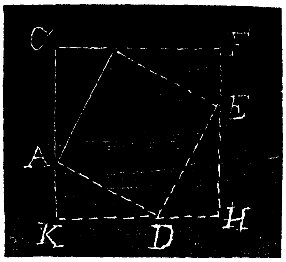


Fig. 40.

NOTE. If we join AE in Fig. 40, and treat the trapezoid $AEFC$ after the manner of the above demonstration, we shall have a slightly different proof. This is known as General Garfield's proof. The writers are indebted, for its reproduction, to Professor Coleman Bancroft, of Hiram College, who gives also the interesting information that the proof was the result of one of Mr. Garfield's *mathematical recreations* while in Congress.

XCV. Fig. 41. Make $BE=BC$, and draw DE perpendicular to AB. Then $DE=DC$.

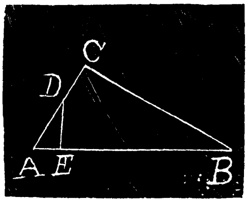


Fig. 41.

Now, the area of $ABC=\frac{1}{2}AC.BC=DE.BC+\frac{1}{2}AE.DE$.

But $DE=(BC.AE)\div AC$, and $AE=AB-BC$.

$\therefore\ \frac{1}{2}AC.BC=\dfrac{BC^2(AB-BC)}{AC}+\frac{1}{2}\dfrac{(AB-BC)^2BC}{AC}$

$\therefore\ AC^2=2AB.BC-2BC^2+AB^2-2AB.BC+BC^2$.

$\therefore\ AB^2=AC^2+BC^2$. Q. E. D.

XCVI. Fig. 41.

Let ADE be the triangle right angled at E. Produce AD to C making $DC=DE$. Produce AE till it meets the perpendicular BC drawn from C.

Then are the two triangles similar.

Now, the area of $ADE=\frac{1}{2}AE.DE=\frac{1}{2}AC.BC-DC.BC$

$$=\frac{1}{2}\frac{DE(AD+DE)^2}{AE}-\frac{DE^2(AD+DE)}{AE}.$$

Whence, $AD^2=AE^2+DE^2$. Q. E. D.

XCVII. Fig. 42.

Let ABC be the triangle, right-angled at C. Draw the escribed circle O, tangent to the hypotenuse.

Designate the sides of the triangle by a, b, c, and the radius of the circle by r.

The area of the square $CPOS=r^2=\frac{1}{2}ab+rc$.

But $r=\frac{1}{2}(a+b+c)$.

$\therefore$ By substitution, $\frac{1}{4}(a+b+c)^2=\frac{1}{2}ab+\frac{1}{2}c(a+b+c)$.

$\therefore\ c^2=a^2+b^2$. Q. E. D.

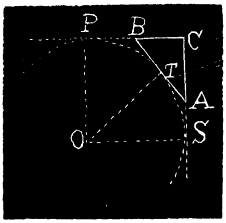
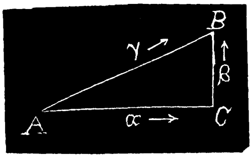


Fig. 42.

IV. QUATERNION PROOFS.

XCVIII. Fig. 43.



Represent the sides as indicated in the figure.

Then $\gamma=\alpha+\beta$. Squaring, $\gamma^2=\alpha^2+2S\alpha\beta+\beta^2$.

But, since angle C is right, $2S\alpha\beta=0$.

$\therefore\ \gamma^2=\alpha^2+\beta^2$, or as lengths simply, changing signs, $AB^2=AC^2+BC^2$. Q. E. D.

Fig. 43.

From Hardy's *Elements of Quaternions*.

XCIX. Fig. 44.

Let ABC be a triangle, right-angled at C. Draw CD perpendicular to CB and equal to AC. Draw DB, which equals AB.

Represent vectors AC and CD by α, and vector CB by β.

$\therefore$ As vectors, $AB=AC+CB=\alpha+\beta$, and $DB=DC+CB=-\alpha+\beta$.

Squaring and adding, we have $AB^2+BD^2=2\alpha^2+2\beta^2$. For the corresponding *lines*, we have $AB^2+BD^2=2AC^2+2CB^2$.

$\therefore AB^2=AC^2+CB^2$. Q. E. D.

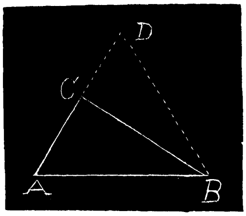


Fig. 44.

C. Fig. 45.

Let ABC be a triangle, right-angled at C. Complete the parallelogram $ADBC$. Draw the diagonal CD.

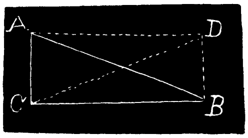


Fig. 45.

Represent vectors CB and AD by α, and vectors CA and BD by β.

$\therefore$ As vectors, $CD=CA+AD=\alpha+\beta$, and $AB=AD+DB=\alpha-\beta$.

Squaring and adding, we have $CD^2+BA^2=2\alpha^2+2\beta^2$. For the corresponding *lines*, we have $CD^2+BA^2=2CA^2+2CB^2$.

$\therefore 2BA^2=2CA^2+2CB^2$.

$\therefore AB^2=AC^2+CB^2$. Q. E. D.

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HISTORICAL NOTE.

BY FLORIAN CAJORI, PH. D., PROFESSOR OF PHYSICS, COLORADO OOLLEGE, AUTHOR OF TWO HISTORIES OF MATHEMATICS.

[Contributed by the request of the authors.]

If the lengths of the sides of a triangle are, respectively, 3, 4, 5 units, then the figure is a right triangle. This fact was known to the early Egyptians, who, it appears, based upon it a method of laying out their temples. They determined a N. and S. line by accurate astronomical observation, then ran a line at right angles to this by means of a rope stretched around three pegs in such a way that the three sides of a triangle thus formed were to each other as 3 : 4 : 5, one of the legs of the right triangle being made to coincide with the N. and S. line.* Essentially the same process was described later by Heron of Alexandria, by the Hindu astronomers, and by Chinese writers. The Hindus took for the lengths of the sides 15, 36, 39, respectively. There is reason to believe that the Egyptian "rope-stretchers" existed as early as the time of King Amenemhat I., about 2300 B. C. If this date is correct, then this method of laying out right angles in the field by rope-stretching was in vogue fully 3000 years!

The discovery of the well-known property of the right triangle is ascribed by Greek writers to Pythagoras. The truth of the theorem for the special case when the sides are 3, 4, 5, respectively, he may have learned from the Egyptians. That the importance and beauty of this theorem of three squares was thoroughly appreciated by the Greeks is evident from the legend to which its discovery gave rise. Pythagoras is said to have been so jubilant over his great achievement, that he offered a hecatomb to the muses who inspired him. As the Pythagoreans believed in the transmigration of the soul and, for that reason, opposed the shedding of blood, the sacrifice was replaced in the traditions of the Neo-Pythagoreans by that of "an ox made of flour"!. The proof given by Pythagoras for this theorem has not been handed down to us. That in *Euclid* I, 47 is due to Euclid himself. Much ingenuity has been expended in conjecture as to the nature of the proof given by Pythagoras. Some critics believe that the proof involved the consideration of special cases; that it was essentially that for the isosceles right triangle outlined by Plato in *Meno*,† in which a square is divided into isosceles right triangles. Other critics surmise that the Pythagorean proof was substantially the same as that given by the Hindu astronomer Bhaskara (about 1150 A. D.), who draws the right triangle four times in the square upon its hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangle. Arranging the small square and the four triangles in a different way, they can be shown, together, to make up the sum of the squares of the two sides. In another place Bhaskara gives a second

*M. Cantor, *Vorlesungen ueber Geschichte der Mathematik*, Vol. I, 1894, page 64.
†Cantor, *op. cit.* page 205.

demonstration of this theorem by drawing from the vertex of the right triangle a perpendicular to the hypotenuse and then suitably manipulating the proportions yielded by the similar triangles. This proof was unknown in Europe until it was rediscovered by the English mathematician, John Wallis.

Among Arabic authors the earliest proof, for the case of the isosceles right triangle, was giveu by Alchwarizmî, who lived in the early part of the 9th century. It is the same as that in Plato's *Meno*. The Persian mathematician, Nasîr Eddîn, who flourished during the early part of the 13th century, gave a new proof, which required the consideration of eight special cases.* Until six years ago this proof was attributed to more recent writers.

The theorem of Pythagoras has received several nicknames. In European universities of the Middle Ages it was called "magister matheseos," because examinations for the degree of A. M. (when held at all) appear usually not to have extended beyond this theorem, which, with its converse, is the last in the first book of Euclid. The name, "pons asinorum," has sometimes been applied to it, though usually this is the sobriquet for *Euclid*, I., 5. Some Arabic writers, Behâ Eddîn for instance, call the Pythagorean theorem, "figure of the bride." Curiously enough, this romantic appellation appears to have originated from a mistranslation of the Greek word νύμφη, applied to the theorem by a Byzantine writer of the 13th century. This Greek word admits of two meanings, "bride" and "winged insect." The figure of the right triangle with the three squares suggests an insect, but Behâ Eddîn apparently translated the word as "bride."†

*See H. Suter in *Bibliotheca Mathematica*, 1892, pages 3 and 4.
†See P. Tannery in *L'Intermediare des Mathematiciens*, 1894, Vol. I, page 254.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

106. Proposed by ELMER SCHUYLER, High Bridge, N. J.

What is the amount of \$1000 at compound interest for three years, at 6%, if it be compounded every instant?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa., and J. OWEN MAHONEY, B. E.. M. Sc., Instructor in Mathematics, Carthage High School, Carthage, Texas.

Let A=amount, P=principal, r=rate, n=number of years, q=number of times interest is payable a year.

Then $A=P[1+(r/q)]^{qn}$. Let $q=rx$.

$\therefore A=P[1+(1/x)^{nrx}=P\{[1+(1/x)]^x\}^{rn}=Pe^{rn}$ when x is infinite.

$\therefore A=1000e^{\frac{9}{50}}=1000\times1.19705=\$1197.05$.

II. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; J. D. CRAIG, Frankfort, Ky.; and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let A be the amount of P dollars for n years at r per cent. payable q times a year. Then $A=P[1+(r/q)]^{nq}=$

$$P\left[1+nr+\frac{n^2r^2}{1.2}-\frac{nr^2}{1.2.q}+\frac{n^3r^3}{1.2.3}+\frac{2n}{1.2.3q^2}-\frac{3n}{1.2.3.q}+\frac{n^4r^4}{1.2.3.4}+\dots\right].$$

Now if interest is to be compounded every instant, q is infinite and hence all terms in this series containing q will vanish, and we have

$$A=P\left[1+nr+\frac{n^2r^2}{1.2}+\frac{n^3r^3}{1.2.3}+\frac{n^4r^4}{1.2.3.4}+\dots\right]=Pe^{nr}.$$

$\therefore$ $A=\$1000(2.71828)^{.18}=\$1197.462+$.

III. Solution by D. G. DORRANCE, Jr., Camden, N. Y.

The formula for the amount of (a) dollars for (n) years at $r\%$ interest compounded every xth part of a year is

$$a\left(1+\frac{r}{x}\right)^{nx}$$

which expanded by the Binomial Theorem becomes

$$a\left(1+nx\frac{r}{x}+\frac{nx(nx-1)}{1.2}\frac{r^2}{x^2}+\frac{nx(nx-1)(nx-2)}{1.2.3}\frac{r^3}{x^3}+\text{ etc.}\right)$$

which, when x is made infinitely large, becomes

$$a\left(1+nr+\frac{n^2r^2}{1.2}+\frac{n^3r^3}{1.2.3}+\frac{n^4r^4}{1.2.3.4}+\frac{n^5r^5}{1.2.3.4.5}+\text{ etc.}\right).$$

Make $a=\$1000.$, $n=3$, and $r=.06$, and the above becomes

$\$1000(1+.18+.0162+.000972+.00004374+.00000157464+\text{etc.})$
$=\$1000(1.19721731464+)=\$1197.21731464+$, the required amount.

Also solved by *CHAS. C. CROSS*, and *ALOIS F. KOVARIK*.

ALGEBRA.

91. Proposed by NELSON S. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.

Solve the following without making use of the determinant notation, and prove that the results obtained are the roots.

$$10x-2y+4z=5,$$
$$3x+5y-3z=7,$$
$$x+3y-2z=2.$$

I. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Eliminate z: $\quad 12x+4y=9 \ldots\ldots(1)$,

$$42x+14y=43 \ldots\ldots(2).$$

Multiplying (1) by 7, and (2) by 2,

$$84x+28y=63,$$
$$84x+28y=86.$$

Whence the values of x, y, and z are infinite.

Multiply (1) by 43, and (2) by 9; equate results and reduce:

$$y=-3x.$$

Similarly, $\quad z=-4x.$

The values $a/0$, $-(3a/0)$, $-(4a/0)$ prove in all three equations.

G. B. M. ZERR, J. K. ELLWOOD, M. A. GRUBER, and *ELMER SCHUYLER* each show that the equations are not simultaneous.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Combining (1) and (3) we find $3x+y=\frac{9}{4} \ldots\ldots(4)$.

Combining (1) and (2) we find $3x+y=\frac{43}{14} \ldots\ldots(5)$.

Combining (2) and (3) we find $3x+y=8 \ldots\ldots(6)$.

These three results are incompatible.

From (1)+(3) we get $15x-19y+18z=0$, or $15(x/z)-19(y/z)=-18$.

From (1)+(3)−(2) we get $8x-4y+5z=0$, or $8(x/z)-4(y/z)=-5$.

Solving these we find the ratios $x/z=-\frac{1}{4}$, $y/z=\frac{3}{4} \ldots\ldots(7)$.

The only answers to (4), (5), or (6) are $x=-\infty$, $y=+\infty$.

Similarly, $z=+\infty$, and the ratio of these infinities is given from (7): $x:y:z::-1:3:4$.

Writing (1), $10(x/z)-2(y/z)+4=5/z=0$, since $z=\infty$.

$\therefore$ $10(-\frac{1}{4})-2(\frac{3}{4})=(-16/4)=-4$, which proves; similarly for (2) and (3).

III. Solution by CHARLES C. CROSS, Libertytown, Md.

Take the general equations,

$$ax+by+cz=d,$$
$$a'x+b'y+c'z=d',$$
$$a''x+b''y+c''z=d''.$$

From which,

$$x=\frac{b'c''d+bc'd''+b''cd'-b'cd''-bc''d'-b''c'd}{ab'c''+a''bc'+a'b''c-a''b'c-a'bc''-ab''c'},$$

$$y=\frac{ac''d'+a'cd''+a''c'd-a''d'c-a'c''d-ad''c'}{D},$$

$$z=\frac{ab'd''+a'b''d+a''bd'-a''b'd-a'bd''-ab''d'}{D}.$$

Whence by substitution $x=\frac{23}{0}$, $y=-\frac{69}{0}$, and $z=-\frac{92}{0}$.

From this we see that a problem is impossible when

$$ab'c''+a''bc'+a'b''c-a''b'c-a'bc''-ab''c'=0.$$

IV. Solution by J. M. BOORMAN, Woodmere, Long Island, N. Y.

(1), (2), or (3) requires a *modulus*, otherwise it lacks one congruous root for some value of x, y, or z. If (1) and (3) be held congruous, modulus (2)$=-$2.875. ∴ Constant (2)$=7+M=4.125$.

$$\left.\begin{array}{l} \therefore x=1.3,\ 1.4,\ \text{etc.};\ 2,\ 2.1\ldots\ldots \\ y=-1.65,\ -1.95,\ \text{etc.};\ -3.75,\ -4.05\ldots\ldots \\ z=-2.825,\ 3.225,\ \text{etc.};\ -5.625,\ -6.925\ldots\ldots \end{array}\right\} \begin{array}{l}\text{The law is obvious.} \\ \\ M=-2.875\ \ldots\ldots(A).\end{array}$$

The above, with modulus (1)$=23$ taking (2) and (3) congruent, modulus (3)$=1.6\frac{3}{7}$. (1) and (2) taken congruent and the infinite series (B) and (C) are the required solution and roots.

PROOF. $\frac{2}{3}(2)$ is $2x+3\frac{1}{3}y=4\frac{2}{3}+2z$; $\frac{1}{2}(1)$ is $5x-y=2.5-2z$. Subtracting (3) from the first of the above and adding it to the second, we have

$$x+\tfrac{1}{3}y=2.66\tfrac{2}{3} \text{ and } 6x+2y=4.5\ldots\ldots(4).$$
$$\underline{x+\tfrac{1}{3}y=0.75} \quad \text{or} \quad x+\tfrac{1}{3}y=0.75.$$

Defect of $2z$ in (2), $-1.91\frac{2}{3}$ of $3z$=modulus (2)$=-2.875$.

If $10x$, $-2y$, etc.$=$; ..... ∴ $y=-3.75$; $z=-5.625$ as above.
(1)$=20+7.5-22.5=5$.
(2)$=3x=6-18.75+16.875=7-M=4.125$.
(3) $x=2-11.25+11.25=2$.
Compute for $x=2.1$ for series (A) and its law.
Similarly, take (2) (3) congruent. ∴ Modulus (1)$=23$. ∴ (B).
Similarly, take (1) (2) congruent. ∴ Modulus (3)$=1\ 6\frac{3}{7}$. ∴ (C).

$$\begin{array}{ll} \therefore x_1=2,\ 2.2,\ 2.4,\ 2.6,\ 2.8,\ \text{etc.} & \text{Modulus }(1)=23. \\ y_1=2.\ 1.4.\ 0.8,\ 0.2,\ -0.4,\ \text{etc.} & \text{Constant }(1)=5+23=28. \\ z_1=3,\ 2.2,\ 1.4,\ 0.8,\ -0.2,\ \text{etc.} & \ldots\ldots(B). \\ x_2=2,\ 2.1.\ 2.2,\ \text{etc.} & \end{array}$$

$$\left.\begin{array}{ll} y_2=-2.9\frac{2}{7},\ -3\ 2\frac{2}{7},\ -3\ 5\frac{2}{7},\ \text{etc.} & \text{Modulus }(3)=1.6\frac{3}{7}. \\ z_2=-5.2\frac{1}{7},\ -5.6\frac{1}{7},\ 6.0\frac{1}{7},\ \text{etc.} & \text{Constant }(3)=36\frac{3}{7}. \end{array}\right\}\ \ldots\ldots\ldots\ (C).$$

Laws of (B) and (C) are obvious.

[NOTE on Problem 89. I have been interested with solutions of the problem "Solve $\left\{\begin{array}{l} x^2+y=7 \\ x+y^2=11 \end{array}\right\}$ by quadratics," the eleventh solution of which appeared in the February number of the MONTHLY.

This example was given me to solve about forty-three years ago, and hardly a year has passed since without having been requested to solve it. A so called quadratic solution appeared in Henkle's "Notes and Queries" about twenty-five years ago; and Professor Schuyler inserted substantially the same solution in the edition of his Algebra of 1884. Several solutions appeared in the mathematical columns of the *New England Journal of Education* about fifteen years ago.

All the solutions in which the values $x=2$ and $y=3$ are found by completing a square are *tentative* and depend indirectly upon the values sought. Take, for example, V. (*b*). $x^2+y=7$ is written $y-3=4-x^2$ and $x+y^2=11$ is written $y^2-9=2-x$. That is, first, y minus the value of y is equal to the square of the value of x minus the square of x; and second, y^2 minus the square of the value of y is equal to the value of x

minus x; and the ability to get an equation of which the square can be completed depends upon this operation,

In the seventh solution Dr. Zerr lets $t=3-y$ and gets $x^2-4=t$ and $x-2=t(3-y)$. Now the only integral value of t that will satisfy this assumption is zero; and this gives $x=2$ and $y=3$. And the the rest of the work is only necessary to show a little algebraic skill in getting a form the square of which can be completed. We are to observe, too, that when the square root is taken the double sign must be excluded.

In regard to the eleventh solution I ask how the solver got the $10ab=40+10b$ to subtract from (5)? I think he will admit that he found it by trial; or if not, he knew the values sought and from them found the member by which (4) must be multiplied to get this result. In every similar example this number is $2(y^2-x^2)(y>x)$. In the present example $2(9-4)=10$. Had the example been $\left\{\begin{array}{l} x^2+y=9 \\ x+y^2=27 \end{array}\right\}$ when $x=2$ and $y=5$, the number with which (4) must be multiplied is $2(25-4)=42$. So by knowing the roots in tnis mode of solution we always can find what to subtract to complete the square. But if we do not know the roots sought the finding of this quantity is a mere work *trial*; and would require much more labor than to find the values by inspection.

So I have come to the conclusion that when there are integral values for x and y they are best found by inspection, and the other values are best found by the use of Sturm's Theorem and Horner's Method of approximation.

M. C. STEVENS.

Purdue University.

This problem, with four different solutions (all of which have been published in the MONTHLY) may be found in the *School Visitor*, Vol. III., pp. 114-115, and the editor, Mr. John S. Royer, calls it the "Yale Problem." Three solutions of this problem may also be found in *The Mathematical Visitor*, Vol. II., p. 3. and two solutions of it appeared in Vol. II., p. 25, of *The Analyst*.

$x^2+y=a$ is the equation of a parabola whose axis coincides with the Y-axis of reference, and its infinite branches extending in the negative direction; $x+y^2=b$ is the equation of a parabola whose axis coincides with the X-axis of reference and its infinite branches extending in the negative direction. These two curves may intersect in four points, intersect in two points and touch in one, intersect in two points, touch in one point or not intersect at all. The equations considered as simultaneous may, therefore, have four real roots, all different; four real roots, two equal and two different; two real roots, equal or different, and two imaginary roots; or four imaginary roots. The solution of the general case leads to a biquadratic which cannot be solved by quadratics.

EDITOR F.]

GEOMETRY.

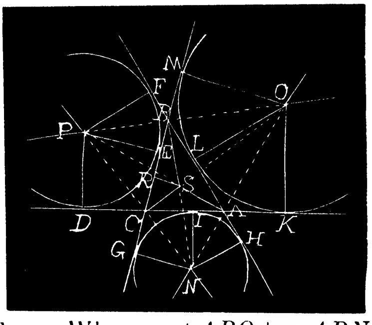
108. Proposed by NELSON L. RORAY, Bridgeton, N. J.

ABC is a triangle. O_1, O_2, O_3 centers of escribed circles. Prove altitudes of triangle $O_1O_2O_3$ are concurrent at center of incribed circle.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take the figure of my solution of Problem 97, Vol. V, No. 10, page 231, THE AMERICAN MATHEMATICAL MONTHLY.

We are to prove BN, AP, and CO the respective altitudes of $\triangle PON$, and passing through S, the center of inscribed circle of $\triangle ABC$.



$\triangle BGN$ and $\triangle BHN$ are equal; for $NG=NH$ (radii of same circle); $\angle BGN=\angle BHN$ =right triangle (radius to point of tangency), and BN is common.

$\therefore$ $\angle ABN=\angle CBN$; and as BN bisects $\angle ABC$, BN passes through S, the center of inscribed circle.

$\angle ABO+\angle ABN+\angle CBN+\angle CBP=$ two right triangles. But $\angle ABO=\angle CBP$ (property of escribed circles), and $\angle ABN=\angle CBN$. $\therefore$ $2\angle ABO=2\angle ABN$=two right angles. Whence $\angle ABO+\angle ABN=\angle NBO$=right triangle.

$\therefore$ BN is perpendicular to PO.

$\therefore$ BN is the altitude of $\triangle PON$ let fall from the vertex N, and passes through the center of the inscribed circle of $\triangle ABC$.

In a similar manner, AP and CO can be proved the other two altitudes passing through S.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

ABC is the pedal triangle of $O_1O_2O_3$.

The altitudes of $O_1O_2O_3$ bisect the angles of its pedal triangle.

$\therefore$ The altitudes are concurrent at the in-center of ABC.

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Using trilinears, and ABC as the triangle of reference, the equation to BC, CA, AB being $\alpha=0$......(1), $\beta=0$......(2), $\gamma=0$......(3), respectively, and noticing that O_1O_2, O_2O_3, O_3O_1 are the external bisectors of angles C, A, B, respectively, we have their equations, in order,

$$\alpha+\beta=0 \ldots\ldots (4),\quad \beta+\gamma=0 \ldots\ldots (5),\quad \gamma+\alpha=0 \ldots\ldots (6).$$

Any line through the intersection of (4) and (5) is given by $\alpha+\beta+k(\beta+\gamma)=0$, or $(1+k)\alpha+\beta+k\gamma=0$......(7).

The condition that (7) is perpendicular to (6) is

$$(1+k)(1-\cos A-\cos B-\cos C)=0 \ldots\ldots (8).$$

This in general requires that $k=-1$......(9), and (7) is $\beta-\gamma=0$...... (10), the internal bisector of $\angle C$. This really proves the theorem.

IV. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.; G. I. HOPKINS, A. M., Professor of Mathematics and Physics, High School, Manchester, N. H.; W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Ia.; MELVIN ENGER, Decorah, Ia.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and J. O. MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Tex.

In the figure of the first solution, change N to O_2, P to O_1, O to O_3, and S to O.

Since COO_3 bisects the angle C, it passes through the centers O of the inscribed circle and O_3 of the escribed circle. Moreover, since CO_1 bisects the supplement of C, O_1CO_3 is a right angle. Hence CO_3 is an altitude of $O_1O_2O_3$. For the same reason, AO_1 as well as BO_2 pass through O and are altitudes of $O_1O_2O_3$.

Excellent demonstrations of this proposition were also furnished by *COOPER D. SCHMITT*, *E. D. SCALES*, *J. SCHEFFER*, and *CHAS. C. CROSS*. Mr. Cross sent in two different demonstrations.

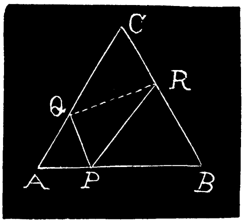
CALCULUS.

83. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

From a given point, P, in the base, AB, of a triangle, to inscribe in the latter the minimum triangle, if its angle at P is given.

I. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

Let PQR be the triangle whose area is to be a minimum. Denote the segments into which P divides the base by a and b and the angles opposite them by θ and φ. Then since the angle PQR is constant $\theta+\varphi$=constant.



Now $PQ=\dfrac{a\sin A}{\sin\theta}$, $PR=\dfrac{b\sin B}{\sin\varphi}$.

In order that the triangle may be a minimum it is evident therefore that $\sin\theta\sin\varphi$ must be a *maximum*.

$\therefore\ \sin\varphi\cos\theta-\sin\theta\cos\varphi=0$.

$\therefore\ \sin(\varphi-\theta)=0$. $\therefore\ \varphi=\theta$. Or if it be required to *construct* the triangle it will be found that $\angle RPB=A+\frac{1}{2}(C-\psi)$ where ψ is the given angle at P.

II. Solution by the PROPOSER.

Let PQR be an inscribed triangle, Q lying in AC and R in BC. Denote the given angle QPR by β, and the given distances PA and PB respectively, by m and n. We have

$$QP=\frac{m\sin A}{\sin(A+\theta)},\quad RP=\frac{n\sin B}{\sin(\beta-B+\theta)}.$$

Since the area of $\triangle APQ$, $\angle QPR$ being constant, depends upon the product of AP and RP, the area is a minimum if $\sin(A+\theta)\sin(\beta-B+\theta)$ is a maximum. Putting this product$=M$, we find

$$\frac{\partial M}{\partial\theta}=\sin(A-B+\beta+2\theta),\ \text{and}\ \frac{\partial^2 M}{\partial\theta^2}=2\cos(A-B+\beta+2\theta).$$

From $\sin(A-B+\beta+2\theta)=0$, we obtain $A-B+\beta+2\theta=180$, since $A-B+\beta+2\theta=0$, would make $\partial^2M/\partial\theta^2$ a positive quantity, and furnish a minimum instead of a maximum.

$\therefore\ \theta=90^\circ-\frac{1}{2}(A-B+\beta)$.

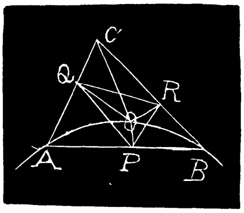
If $\beta=C$, $\theta=B$, and in this case the minimum triangle would have its sides QP and RP parallel to BC and AC respectively.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and ELMER SCHUYLER, High Bridge, N. J.

$\angle APQ=\theta$, $\angle QPR=\lambda$.

$\therefore\ PQ=d\sin A\operatorname{cosec}(A+\theta)$, $PR=(c-d)\sin B\operatorname{cosec}(\lambda+\theta-B)$.

$\therefore\ \operatorname{cosec}(A+\theta)\operatorname{cosec}(\lambda+\theta-B)$=minimum.



This is the case when $\cot(A+\theta)+\cot(\lambda+\theta-B)=0$.

$\therefore\ \theta=\frac{1}{2}\pi-\frac{1}{2}(A+\lambda-B)$.

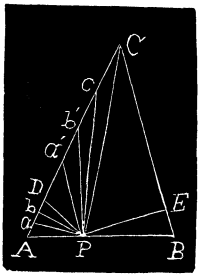
$\therefore\ PQ=d\sin A\operatorname{cosec}\frac{1}{2}(\lambda+C)$.

$PR=(c-d)\sin B\operatorname{cosec}\frac{1}{2}(\lambda+C$.

Geometrical construction : Upon AB describe a segment containing an angle$=\angle C+\angle\lambda$. At P erect PO perpendicular to AB meeting the circle in O ; draw OR, OQ perpendicular to BC, AC, respectively. Then PQR is the minimum triangle required.

IV. Solution by P. H. PHILBRICK, C.E., Chief Engineer, Kansas City, Watkins & Gulf Railway, Lake Charles, La.

Let the figure represent the triangle. Let angle at $P=2\theta$. Drop the perpendicular PD upon the nearest side, then draw Pa and Pa' making angles DPa and DPa' each equal to θ. Paa' is the triangle required.



To prove that this triangle is less than any other inscribed triangle having the same vertex angle at P, draw Pb and Pb' making angles aPb and $a'Pb'$ each$=x$.

Let $PD=p$. Now $aa'=2p\tan\theta$.

Also $bb'=bD+Db'=p[\tan(\theta-x)+\tan(\theta+x)]$.

$$\tan(\theta-x)=\frac{\tan\theta-\tan x}{1+\tan\theta\tan x} \text{ and } \tan(\theta+x)=\frac{\tan\theta+\tan x}{1-\tan\theta\tan x}.$$

Substituting and reducing, we find,

$$bb'=2p\tan\theta\times\frac{1+\tan^2 x}{1-\tan^2\theta\tan^2 x}>2p\tan\theta.$$

$\therefore\ bb'>aa'$ and triangle $Pbb'>$triangle Paa'.

SECOND PROOF.

$$\tan(\theta-x)=(\theta-x)+\frac{(\theta-x)^3}{3}+\ldots\ldots, \text{ and } \tan(\theta+x)=(\theta+x)+\frac{(\theta+x)^3}{3}+\ \ldots\ldots;$$

$$\text{also } 2\tan\theta=2\theta+2\frac{\theta^3}{3}+\ldots\ldots$$

But each term in the last development, except the first, is less than the sum of the corresponding terms in the first two developments.

Hence $\tan(\theta-x)+\tan(\theta+x)>2\tan\theta$. Hence $bb'>aa'$.

A geometrical proof is also easy.

In case half the angle at P is $>APD$, the triangle, as above, would not lie entirely within ABC. In that case make angle $APc=$the given angle 2θ, and APc is the triangle required.

The maximum triangle is formed by drawing from P a line PC to the farther vertex of the farthest side and making $CPE=2\theta$. If E falls on CB prolonged, then CPB is the required triangle. It will be observed that if $PB>PA$ triangle $CPB>CPA$.

MECHANICS.

75. Proposed by B. F. FINKEL, A.M.,M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A particle, P, is held in a bent tube by two forces directed towards two fixed points, H and S. Show that the equation of the tube is $PS.PH=k^2$, if the forces are μ/PS and μ/PH.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Let $PS=r$, $PH=r'$, $\mu/PS=f$, $\mu/PH=f'$.

By the principle of virtual work we have for equilibrium,

$$fdr+f'dr'=0, \text{ but } f/f'=r'/r \text{ or } f'r'=fr.$$

Dividing $fdr=-f'dr'$ by $fr=f'r'$ we get

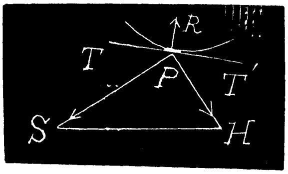
$$dr/r=-dr'/r' \text{ or } rdr'+r'dr=0.$$

Integrating, we get $rr'=$a constant$=k^2$.

$\therefore PS.PH=k^2$.

IV. Solution by GEORGE LILLEY, Ph.D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

Let P be any position of the particle, TT' the tangent to the tube at P, $\angle TPS=\phi$, $\angle TPH=\phi'$, $\angle PSH=\theta$, $\angle PHS=\theta'$, $PS=r$ and $PH=r'$.



Resolve along TT', $\dfrac{\mu}{r}\cos\phi+\dfrac{\mu}{r'}\cos\phi'=0$.

But, $\tan\phi=r\dfrac{d\theta}{dr}$; hence, $\cos\phi=\dfrac{dr}{ds}$, where ds is element length of the tube. Also $\cos\phi'=\dfrac{dr}{ds}$.

Therefore, $\dfrac{dr}{2r}+\dfrac{dr'}{2r'}=0$.

Integrating, $\log\sqrt{(rr')}+c=0$, where c depends on known values of r and r'.

Therefore, $SP.PH=k^2$.

Or thus : By the method of virtual work,

$Fdr+F'dr'=0$, where $F=\dfrac{\mu}{SP}$ and $F'=\dfrac{\mu}{HP}$.

Thus form the differential equation and solve it as above.

V. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

Let P be the particle, S and H the fixed points. Denote PS, PH, and SH by h, s, and $2m$, respectively. Take the origin of rectangular axes midway between S and H, the x-axis lying along the line SH. Let the resultant force

acting at P (x, y) intersect Ox at A, and make angles α, β, θ, with PS, PH, and AH, respectively.

Since $\dfrac{\mu}{SP}:\dfrac{\mu}{HP}=\sin\beta:\sin\alpha$, $HP:SP=\sin\beta:\sin\alpha$,

$$\text{or } \frac{\sin\alpha}{\sin\beta}=\frac{h}{s} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1).}$$

Also $\dfrac{s.\sin\beta}{h.\sin\alpha}=\dfrac{AH}{AS}=\dfrac{m-OA}{m+OA}$.

Substitute from (1) and solve for OA, obtaining

$$OA=\frac{h^2-s^2}{h^2+s^2}m. \quad \text{Tan}\theta=\frac{y}{x-OA}=\frac{y}{x-\dfrac{h^2-s^2}{h^2+s^2}m}=\frac{y^3+x^2y+m^2y}{x^3+xy^2-m^2x}.$$

Since PA is normal to the tube, the differential equation of the curve is

$$\frac{dy}{dx}=-\frac{x^3+xy^2-m^2x}{y^3+x^2y+m^2y}.$$

Integrating, $y^4+x^4+2x^2y^2+2m^2y^2-2m^2x^2=c$.
Adding m^4 to both members, and factoring,

$$[y^2+(m+x)^2][y^2+(m-x)^2]=c+m^4, \text{ or } SP^2.HP^2=c+m^4=k^4,$$

from which $SP.HP=k^2$.

VI. Solution by R. E. GAINES, Professor of Mathematics, Richmond College, Richmond, Va.

Denote PS and PH by r and r', respectively, and these may be taken as the "bipunctual coördinates" of P. Then it is easy to show that

$$\frac{dr'}{ds}=\cos\psi \text{ and } \frac{dr}{ds}=-\cos\varphi.$$

$$\therefore \frac{dr'}{dr}=-\frac{\cos\psi}{\cos\varphi}.$$

Now resolving forces along the tangent at P we have,

$$\frac{\mu}{r}\cos\varphi=\frac{\mu}{r'}\cos\psi \text{ or } \frac{1}{r}-\frac{1}{r'}\frac{\cos\psi}{\cos\varphi}=0.$$

$$\therefore \frac{1}{r}+\frac{1}{r'}\frac{dr'}{dr}=0.$$

$$\therefore \log r+\log r'=\log k^2. \quad \therefore rr'=k^2.$$

If the forces had been $\mu f(r)$ and $\mu F(r')$ we could get the form of the curve by integrating

$$f(r)+F(r')\frac{dr'}{dr}=0.$$

76. Proposed by JAMES F. LAWRENCE, Classical Sophomore, Drury College, Springfield, Mo.

An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the incline plane, x the part of the plane rolled over by the sphere, prove that $(M+m)x'=mx\cos\alpha$, $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$, where α is the inclination of the plane. [From *Routh's Elementary Rigid Dynamics*, page 126.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Let F=friction of the sphere and plane, R=their mutual reaction, θ=the angle through which the sphere has rotated from the beginning of motion, y=the vertical distance of the center of the sphere from the horizontal plane, x_1=the corresponding abscissa, h and k the initial values of x and y, respectively, and a= the radius of the sphere.

For the motion of the sphere, resolving horizontally and vertically, and taking moments about the center of the sphere,

$$m\frac{d^2x_1}{dt^2}=F\cos\alpha-R\sin\alpha\,\ldots\ldots..(1),\quad m\frac{d^2y}{dt^2}=F\sin\alpha+R\cos\alpha-mg\,\ldots\ldots..(2),$$

$$\tfrac{2}{5}ma^2\frac{d^2\theta}{dt^2}=aF\,\ldots\ldots..(3).$$

For the horizontal motion of the plane,

$$M\frac{d^2x'}{dt^2}=-F\cos\alpha+R\sin\alpha\,\ldots\ldots..(4).$$

Also, $x_1=h+x'-a\theta\cos\alpha$...............(5), $y=k-a\theta\sin\alpha$...............(6).

From (5) and (1), $m\dfrac{d^2x'}{dt^2}-ma\cos\alpha\dfrac{d^2\theta}{dt^2}=F\cos\alpha-R\sin\alpha$...........(7);

and from (6) and (2), $-ma\sin\alpha\dfrac{d^2\theta}{dt^2}=F\sin\alpha+R\cos\alpha-mg$..............(8).

Eliminating F and R from (3), (7) and (8),

$$m\cos\alpha\frac{d^2x'}{dt^2}=\tfrac{7}{5}ma\frac{d^2\theta}{dt^2}-mg\sin\alpha\,\ldots..(9).$$

Integrating (9), noticing that when $t=0$, $\dfrac{dx^2}{dt}=0$, $\dfrac{d\theta}{dt}=0$, and $x'=0$, $\theta=0$,

$$m\cos\alpha.x' = \tfrac{7}{5}ma\theta - \tfrac{1}{2}mg\sin\alpha.t^2 \quad .....(10).$$

Again, eliminating F and R from (1) and (4),

$$m\frac{d^2x_1}{dt^2} + M\frac{d^2x'}{dt^2} = 0...(11). \quad \text{But from (5), } m\frac{d^2x_1}{dt^2} = -ma\cos\alpha\frac{d^2\theta}{dt^2}...(12).$$

$$(11)-(12) \text{ gives } ma\theta = \frac{(M+m)x'}{\cos\alpha}......(13).$$

$$(13) \text{ in } (10) \text{ gives } x' = \frac{5m\sin\alpha\cos\alpha}{7(M+m)-5m\cos^2\alpha}\cdot\frac{gt^2}{2}......(14).$$

$$(14) \text{ in } (13) \text{ gives } a\theta = \frac{5(M+m)\sin\alpha}{7(M+m)-5m\cos^2\alpha} \quad .....(15).$$

Since $x = a\theta$, $\frac{7}{5}x - x'\cos\alpha = \frac{1}{2}gt^2\sin\alpha$..(16), and (13) is $(M+m)x' = mx\cos\alpha$..(17).

II. **Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, Univeriity of Mississippi, P. O., University, Miss.**

The horizontal component of the mutual action between the sphere and the inclined plane imparts to M the acceleration $2x'/t^2$ and to m the acceleration $\frac{2(x\cos\alpha - x')}{t^2}$. The forces producing these opposite motions being equal,

$$M\frac{2x'}{t^2} = m\frac{2(x\cos\alpha - x')}{t^2}.$$

From this $(M+m)x' = mx\cos\alpha$ ..... (1).
The principal of *vis viva* gives

$$2mgx\sin\alpha = M\frac{4x'^2}{t^2} + m\frac{4(x\cos\alpha - x')^2}{t^2} + m\frac{4x^2\sin^2\alpha}{t^2} + mk^2\left(\frac{d\theta}{dt}\right)^2$$

in which the first member is twice the work done by gravity, the first term of the second member has reference to the plane, the second term to the horizontal motion of the sphere, the third to its vertical motion, and the last to its rotation, $d\theta/dt$ being its angular velocity, and k^2 its radius of gyration about a diameter.

The motion of the sphere down the plane being one of pure rolling, $a(d\theta/dt) = 2x/t$. (a=radius of sphere).

Substitute $2x/at$ for $d\theta/dt$, and $\frac{2}{5}a^2$ for k^2, and reduce, obtaining

$$\tfrac{1}{2}mgt^2x\sin\alpha = (M+m)x'^2 + \tfrac{7}{5}mx^2 - 2mxx'\cos\alpha.$$

Using (1), this reduces to $\frac{1}{2}gt^2\sin\alpha = \frac{7}{5}x - x'\cos\alpha$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and CHARLES E. MYERS, Canton, O.

Take A as origin. Let (k, l), (y, z) be the coödinates of the center of the sphere when $t=0$ and when $t=t$, respectively, R, the reaction of the plane, P, the friction, a, the radius of the sphere, $\angle DOE=\theta$. Then $AA'=x'$ and $a\theta=x$.

For the sphere the equations of motion are

$\frac{2}{5}ma^2(d^2\theta/dt^2)=aP$..........(1). $m(d^2y/dt^2)=P\cos a-R\sin a$...........(2).

$$m(d^2z/dt^2)=P\sin a+R\cos a-mg\text{..........(3)}.$$

For the plane, $M(d^2x'/dt^2)=-P\cos a+R\sin a$......................(4).

By geometry, $y=k+x'-a\theta\cos a$, $z=l-a\theta\sin a$...................(5, 6).

From (5) and (6) we get $d^2y/dt^2=d^2x'/dt^2-a\cos a(d^2\theta/dt^2)$........(7).

$$d^2z/dt^2=-a\sin a(d^2\theta/dt^2)\text{........(8)}.$$

(7)$\cos a$+(8)$\sin a$ gives

$\cos a(d^2y/dt^2)+\sin a(d^2z/dt^2)=\cos a(d^2x'/dt^2)-a(d^2\theta/dt^2)$...... ....(9).

(2)$\cos a$+(3)$\sin a$ gives $m\cos a(d^2y/dt^2)+m\sin a(d^2x/dt^2)=P-mg\sin a$......(10).

From (9) and (10), $m\cos a(d^2x'/dt^2)-ma(d^2\theta/dt^2)=P-mg\sin a$.....(11).

(2)+(4) gives $m(d^2y/dt^2)+M(d^2x'/dt^2)=0$................. .....(12).

From (12) and (7) we get $(M+m)(d^2x'/dt^2=ma\cos a(d^2\theta/dt^2)$... ..(13).

The value of P from (1) in (11) gives

$$\tfrac{7}{5}a(d^2\theta/dt^2)-\cos a(d^2x'/dt^2)=g\sin a\text{......(14)}.$$

Integrating (13) and (14) and remembering that when $t=0$, $x=0$ and $\theta=0$ we get

$$(M+m)x'=ma\theta\cos a \text{ or } (M+m)x'=mx\cos a,$$

$$\tfrac{7}{5}a\theta-x'\cos a=\tfrac{1}{2}gt\sin a \text{ or } \tfrac{7}{5}x-x'\cos a=\tfrac{1}{2}gt^2\sin a.$$

DIOPHANTINE ANALYSIS.

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71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

I. Solution by CHARLES C. CROSS, Libertytown, Md.

On page 301, Vol. V, of MONTHLY I found four general numbers to be: m, $n^2-1+(m-1)(n-1)^2$, $n(mn+2)$, and $4m(mn^2-mn+2n-1)^2+4(mn^2-mn+2n-1)$.

If we take m and m for the first two numbers, then $m.m+1=\square=y^2$(say) $=(y-s)^2=y^2-2sy+s^2$.

$\therefore y=(s^2+1)/2s$; whence $m=(s^2-1)/2s$.

Let $s=2$, then $m=\frac{3}{4}$. Take $n=2$ and the numbers are $\frac{3}{4}$, $\frac{3}{4}$, $\frac{11}{4}$, 7, and $\frac{315}{4}$.

And the five numbers are $xy=l^2-1$, x, y, $2l+x+y$, $4l(l+x)(l+y)$, and $v=\dfrac{2r+2p(s+1)}{(s-1)^2}$.

$r=xyz+xyu+xzu+yzu$; $s=xyzu$ and $p=x+y+z+u$.

If $l^2=4$ then the five numbers $=1$, 3, 8, 120, and $\dfrac{777480}{(2879)^2}$.

If $l^2=25$ then the five numbers $=1$, 24, 35, 3480, not carried out.
2, 12, 24, 2380, not carried out.
3, 8, 21, 2080, not carried out.
4, 6, 20, 1980, $\dfrac{3822388020}{(950399)^2}$.

The Hillsboro Mathematical Club have solved for an integral value by extending the series, without result. A. H. Bell.

AVERAGE AND PROBABILITY.

65. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

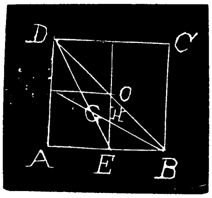
What is the average rate of the sun's motion in declination from the equator to the solstices ?

No solution of this problem has been received.

66. Proposed by REV. W. ALLEN WHITWORTH, A. M.

A rod 9 feet long is to be divided into three parts, of which A is to have the largest, B the next, and C the smallest. If the two fractures are made at random, A's, B's, and C's expectations will be, respectively, 66, 30, and 12 inches. But, if one fracture be made at random and the larger portion of the rod be then divided at random, their expectations will be 64, 31, and 13 inches.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.



Let $AB=AD=108=a$, $AE=AF=\frac{1}{2}a$.

I. To find the mean value of the least part, we find the values or limits of x, y by restricting the point to the area GOH. The limits for the denominator are given by restricting the point to GOB.

$$\therefore L=\frac{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{2}(a-x)}^{x} x\,dx\,dy}{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{2}(a-x)}^{x} dx\,dy+\int_{\frac{1}{2}a}^{a}\int_{\frac{1}{2}(a-x)}^{a-x} dx\,dy}=\frac{\frac{a^3}{108}}{\frac{a^2}{12}}=\tfrac{1}{9}a=12 \text{ inches.}$$

For the mean value of the greatest part the limits are the same for both integrations :

$$\therefore G=\frac{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{2}(a-x)}^{x} xdxdy+\int_{\frac{1}{2}a}^{a}\int_{\frac{1}{2}(a-x)}^{a-x} xdxdy}{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{2}(a-x)}^{x} dxdy+\int_{\frac{1}{2}a}^{a}\int_{\frac{1}{2}(a-x)}^{a-x} dxdy}=\frac{\frac{11a^3}{216}}{\frac{a^2}{12}}=\tfrac{11}{18}a=66 \text{ inches.}$$

$\therefore M=a-(\frac{1}{9}a+\frac{11}{18}a)=\frac{5}{18}a=30$ inches.

II. For the mean value of the least part, the limits for x are found by restricting the point to the area ADG, for y, by restricting the point to $AEHG$ and doubling.

$$\therefore L=\frac{\int_{0}^{\frac{1}{3}a}\int_{x}^{a-2x} xdxdy+2\left[\int_{0}^{\frac{1}{3}a}\int_{y}^{\frac{1}{2}(a-y)} ydydx+\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{a-2x}^{\frac{1}{2}(a-x)} ydxdy\right]}{\int_{0}^{\frac{1}{2}a}\int_{0}^{a-x} dxdy},$$

$$=\left[\frac{a^3}{54}+\frac{a^3}{54}+\frac{7a^3}{864}\right]\div\frac{3a^2}{8}=\frac{13a}{108}=13 \text{ inches.}$$

For the mean value of the greatest part, x is given by restricting the point to GOE, y, by $DFGO$ and doubling.

$$\therefore G=\frac{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{a-2x}^{x} xdxdy+2\left[\int_{\frac{1}{2}a}^{a}\int_{0}^{a-y} ydydx+\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{a-2y}^{y} ydydx\right]}{\int_{0}^{\frac{1}{2}a}\int_{0}^{a-x} dxdy},$$

$$=\left[\frac{a^3}{54}+\frac{a^3}{6}+\frac{a^3}{27}\right]\div\frac{3a^2}{8}=\frac{16a}{27}=64 \text{ inches.}$$

$$\therefore M=a-\left[\frac{13a}{108}+\frac{16a}{27}\right]=\frac{31a}{108}=31 \text{ inches.}$$

67. **Proposed by B. F. FINKEL, A.M.,M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A person writes n letters and addresses n envelopes; if the letters are placed in the envelopes at random, what is the probability that every letter goes wrong? [From *Hall and Knight's Higher Algebra.*]

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let a, b, c, ...n represent the letters.

The number of all possible cases is $1.2.3\ldots\ldots n=n\,!=u_n$ say.

The number of cases, the first letter a, and envelope used is u_{n-1}.

The number of cases, the second letter and envelope used (without the first a) is $u_{n-1}-u_{n-2}$.

The number of cases, the third letter and envelope used (without the first having been a or the second b) is $u_{n-1}-2u_{n-2}+u_{n-3}$.

Thus we have successively,

$$\begin{aligned}&u_{n-1},\\&u_{n-1}-u_{n-2},\\&u_{n-1}-2u_{n-2}+u_{n-3},\\&u_{n-1}-3u_{n-2}+3u_{n-3}-u_{n-4},\\&\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

The sum of these n lines is

$$S=u_n\left[\frac{1}{1}-\frac{1}{2\,!}+\frac{1}{3\,!}-\frac{1}{4\,!}+\ldots\ldots-\frac{(-1)^n}{n\,!}\right].$$

C, the required chance, is $1-S/u_n$.

$$\therefore\ C=\frac{1}{2\,!}-\frac{1}{3\,!}+\frac{1}{4\,!}-\ldots\ldots+\frac{(-1)^n}{n\,!}.$$

[NOTE. A solution of problem 61 will be published in the next issue. If any one will send us a solution of problem 65, it will also appear in the next number. ED. F.]

67a. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A witness in court who undertook to recognize the signature of an individual failed four times in succession. What is the probability that he was correct the fifth time? An actual occurance.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let p represent the probability of his failing again the fifth time, then we have,

$$p=\int_0^1 x^5\,dx\div\int_0^1 x^4\,dx=\tfrac{5}{6}.$$

$\therefore$ The probability of his being correct the fifth time is $\frac{1}{6}$.

II. Solution by the PROPOSER.

In the absence of other evidence we are compelled to judge of the credibility of the witness by his present testimony.

If x represents his credibility, the *apriori* probability that he would have testified falsely four times in succession is $(1-x)^4$.

Hence the chance that x had any particular value is $(1-x)^4dx/\int_0^1(1-x)^4dx$ $=5(1-x)^4dx$, and the chance of the event if x has this particular value is x.

Hence the chance of the event through any particular value of x is $5x(1-x)^4dx$ and the chance through all possible values is $\int_0^1 5x(1-x)^4dx=\frac{1}{6}$.

That is, the chances are 5 to 1 against the correctness of his testimony.

MISCELLANEOUS.

68. Proposed by **WILLIAM HOOVER, A. M., Ph. D.,** Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Find the locus of the vertex of the cone enveloping the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ so that the plane of contact will constantly touch $x^2+y^2+z^2=r^2$.

I. Solution by **ELMER SCHUYLER,** High Bridge, N. J.

From Aldis's *Solid Geometry*, equation of plane of contact to ellipsoid is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1=0,\ \frac{xx'}{r^2}+\frac{yy'}{r^2}+\frac{zz'}{r^2}-1=0,$$

equation of tangent to circle, $x'^2+y'^2+z'^2=r^2$.

$\therefore$ Since the plane of contact is tangent to sphere,

$$\frac{x'}{r^2}=\frac{\alpha}{a^2},\quad \frac{y'}{r^2}=\frac{\beta}{b^2},\quad \frac{z'}{r^2}=\frac{\gamma}{c^2} \text{ and } \left(\frac{\alpha r^2}{a^2}\right)^2+\left(\frac{\beta r^2}{b^2}\right)^2+\left(\frac{\gamma r^2}{c^2}\right)^2=r^2,$$

or equation is

$$\left(\frac{\alpha r}{a^2}\right)^2+\left(\frac{\beta r}{b^2}\right)^2+\left(\frac{\gamma r}{c^2}\right)^2=1,$$

which locus is an ellipsoid similar to the given one but with ratio of axes as $\sqrt{r}:1$.

II. Solution by **W. B. CARVER,** Senior Class, Dickinson College, Carlisle, Pa.

Let (x', y', z') be a point. Then $b^2c^2x'x+a^2c^2y'y+a^2b^2z'z=a^2b^2c^2$ is the equation of the plane of contact of the cone whose vertex is at (x', y', z') with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The condition that this plane touch the sphere $x^2+y^2+z^2=r^2$ is

$$b^4c^4x'^2+a^4c^4y'^2+a^4b^4z'^2=\frac{a^4b^4c^4}{r^2}$$

Letting (x', y', z') move and x', y', and z' become variables, we have for the required locus

$$b^4c^4x^2+a^4c^4y^2+a^4b^4z^2=\frac{a^4b^4c^4}{r^2} \text{ or } \frac{x^2}{a^4/r^2}+\frac{y^2}{b^4/r^2}+\frac{z^2}{c^4/r^2}=1,$$

which is the equation of an ellipsoid.

III. Solution by the **PROPOSER.**

If (x', y', z') be the vertex of the cone, the plane of contact is

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}-1=0\ldots\ldots(1);$$

and the sphere being $x^2+y^2+z^2-r^2=0$......(2), the condition that (2) is touched by (1) is

$$\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}-\frac{1}{r^2}=0\ldots\ldots(3).$$

a concentric ellipsoid.

69. **Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, P. O., Sebastopol, Cal.**

Find the locus of a point equidistant from the circumferences of two fixed circles.

Solution by ELMER SCHUYLER, High Bridge, N. J.

Let radii be a and b, and $OO'=c$.

$OP=\sqrt{(x^2+y^2)}$............(1), $O'P=\sqrt{[y^2+(c-x)^2]}$...... .....(2).

By condition, $O'P-OP=b-a$.

$\therefore \sqrt{[(c-x)^2+y^2]}-\sqrt{[x^2+y^2]}=b-a$............(3).

Clearing of fractions gives us $[c^2-(b-a)^2-2cx]^2=4(b-a)^2(x^2+y^2)$..(4).

This is a conic section and an ellipse, hyperbola, parabola (or particular case) according as $(b-a)^2[(b-a)^2-c^2]>$, $<$, or $=0$.

NOTE ON RIGHT TRIANGLES.

Every right-angled triangle has two concealed roots. By three different combinations of the two roots, the three sides are formed. The longest side is the sum of the squares of the two roots. The second side is the difference of the squares of the two roots. The third side is twice the product of the two roots.

The perimeter is equal to twice the greater root multiplied by the sum of the two roots. The area is equal to the product of the two roots multiplied by the product of the sum and difference of the two roots.

A prime right-angled triangle is one whose sides are integral and cannot all be divided by the same number without a remainder. A prime triangle is the result of having one of the roots odd and one even. Exception.—If the even root is just twice the odd root, the resulting triangle will not be prime, as its sides will all be divisible by the square of the odd root.

If both the roots be odd or both even the sides of the triangle will be divisible by two and the triangle will not be prime. Any odd number may be one side of a prime triangle; in many cases the same odd number will serve as one side of several different prime triangles, as for example, 13, 12, 5, and 13, 84, 85. Any even number divisible by 4 can be one side of a prime triangle as a prime triangle has always one even side. The digit, 2, must be a factor twice, or both the roots will be odd numbers. The same even number may be a side of several prime triangles, as 12, 13, 5, and 12, 35, 37.

A prime right-angled triangle has two of its sides expressed by odd numbers. Find the sum and the difference of these. Each will be the double of a perfect square. The square root of one-half the sum will be the greater root of

the triangle, and the square root of one-half the difference will be the lesser root of the triangle.

Next take the hypotenuse and the even side of any prime right-angled triangle. The sum and the difference will each be a perfect square number, and their square roots will be the sum and the difference of the two roots of the triangle.

A given area, or a given perimeter, can belong to but one prime right-angled triangle, and either the area or the perimeter being given it is easy to find the other dimensions. In case of one side only being given it has been shown that the given side may belong to several different triangles, but all of them are easily found.

C. W. SHEDD.

Columbus, Miss.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

110. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

By measuring with a yard $m=12\frac{1}{2}\%$ *too short*, my profits are $n=25\%$ of my sales. If my yard be $p=10\%$ *too long*, what per cent. of my sales will be my profits ?

111. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

By what per cent. of its original dimensions must a linear yard of steel rail, weighing 60 pounds, be increased so that it may weigh 75 pounds ?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than May 10.

ALGEBRA.

98. Proposed by B. F. FINKEL, A.M.,M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A and B agreed to reap a field of grain for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days; but B, who did not work as quickly as he expected, was obliged to call to his assistance C, an inferior workman, who worked the last two days, in consequence of which B received 3*s*. 9*d*. less than would otherwise have been due him. In what time could B and C each reap the field ? From *Milne's High School Algebra.*

99. Proposed by C. H. JUDSON, Greenville, S. C.

Seven persons met at a summer resort, and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated.

⁂ Solutions of this problem should be sent to J. M. Colaw not later than May 10.

GEOMETRY.

118. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

A picture b feet long hangs on a wall at an inclination θ to the wall, with its base a feet from the floor. How far from the wall should an admirer sit to see it to the best advantage, supposing the light to be equally distributed throughout the room ?

119. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

A sphere touches each of two straight lines which are inclined to each other at a right angle but do not meet; show that the locus of its center is an hyperbolic paraboloid.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

CALCULUS.

88. Proposed by JOHN M. ARNOLD, Crompton, R. I.

When a watch is wound up, the mainspring is closely coiled around a cylindrical piece called the hub of the barrel-arbor. When entirely run down the spring forms an annulus against the inner circumference of the barrel. Show that if the width of the annnlus is a little more than one-fourth of the radius of the barrel, the spring will run the watch the greatest number of hours at one winding, the diameter of the hub being one-third the inside diameter of the barrel.

89. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Integrate the equation,

$$\frac{dy}{dx}+y\cos x=\frac{\sin 2x}{2}.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

MECHANICS.

87. Proposed by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

"He on his impious foes onward drove,
Drove them before him to the bounds
And crystal walls of Heaven; which opening wide
Rolled inward and a spacious gap disclosed
Into the wasteful deep; headlong themselves they threw
Down from the verge of Heaven.
Nine days they fell; Hell at last
Yawning received them whole and on them closed."

Paradise Lost, Book VI.

Assuming Hell to be the center of the earth and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

88. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Show that the equation to the trajectory is

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha},$$

and that v and a can be varied at pleasure, the projectile can in general be made to traverse any two given points in the same vertical plane with the point of projection. [Ex. 83, page 244, Deschanel's *Natural Philosophy*, Part I.]

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

MISCELLANEOUS.

75. Proposed by J. C. NAGLE, Professor of Civil Engineering, Agricultural and Mechanical College, College Station, Texas.

The water tank at the Nacogdoches River on the H. E. & W. T. Ry. is filled by a 3-inch pipe from a reservoir in which the water level is 6 feet above water in tank when full. The top diameter of tank is 17 feet, the bottom diameter is 19 feet, 8 inches, and the pipe projects 10 inches through the bottom. The depth is 13 feet, 6 inches. Find the time required to fill tank, taking the pipe as clean and free from sharp bends, except the right-angled one directly under tank. This bend is 12 feet below outlet of pipe, so that the total length of pipe is 1972 feet. Compare the result with the time of filling if the inlet pipe projected over top of tank instead of entering at the bottom.

76. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Show that

$$\log[x-a-b\surd(-1)]=\tfrac{1}{2}\log[(x-a)^2+b^2]-\surd(-1)\tan^{-1}\frac{b}{x-a},$$

Naperian logarithms being used.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

EDITORIALS.

Contributors are requested to send in select problems for the Departments of Arithmetic, Mechanics, and Average and Probability.

The *Annals of Mathematics*, published for the past fifteen years at the University of Virginia, is to be transferred to Harvard University with the close of the present volume.

Mathematics and the mathematical world have sustained a great loss in the death of Prof. Sophus Lie, which occured at Christiania, Monday, February 18. He was Professor of Geometry at the University of Leipzig from 1886 to 1898, and at the time of his death was Professor of Mathematics at the University of Christiania. We hope to be able to give a biographical sketch of Professor Lie in a future issue of the MONTHLY.

Cornell University announces a Course of Instruction during the Summer, session to be held July 5 to August 16. The following courses in Advanced Mathematics are offered : Advanced Integral Calculus, Prof. Wait ; Differential Equations, Dr. Murray ; Projective Geometry, Prof. Wait ; Theory of Functions of a Complex Variable, Dr. G. A. Miller ; Theory of Groups of a Finite Order, Dr. G. A. Miller ; Theory of Numbers, Dr. G. A. Miller.

BOOKS AND PERIODICALS.

Tables of Logarithms to Five Places of Decimals with Auxiliary Tables. Edited by Edwin S. Crawley, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania. 8vo. Cloth, xxxii+76 pages. Price, 75 cents. Published by the author.

These tables have been prepared to accompany Dr. Crawley's *Elements of Trigonometry* and we are told by the author that great care has been taken to secure accuracy.—the proof having been compared twice, number by number, with different standard tables, *e. g.* Vega's seven-place tables, seventy-fourth edition edited by W. L. Fischer, and Gauss's five-place tables, twentieth edition. In addition to this careful comparison, the method of differences was also applied as a further check. A table not usually found in a collection of tables of this sort, and one which is quite useful in many lines of mathematical work, is table VI in which every degree, minute and second from 0 to 180° is expressed in Radians. B. F. F.

A Short History of Astronomy. By Arthur Berry, M. A., Fellow and Assistant Tutor of King's College, Cambridge ; Fellow of University College, London. 8vo. Cloth, xxxii+440 pages. Price, $1.50. New York : Charles Scribner's Sons.

In this work we have an interesting account of the progress and development of Astromomy from the earliest time down to the present day, and presented in a form intelligible to readers having no special knowledge of either mathematics or Astronomy. Only the essence is here preserved. Little mention is made of Astronomical Instruments, it being held by the author that little pleasure or profit is derived from a written description of scientific Instruments. A most valuable aud interesting feature of the work is the short biographical sketches of leading Astronomers, accompanied by an excellent portrait printed on heavy paper. Of the biographical sketches accompanied by portraits we mention Copernicus, Tycho Brahe, Galilei, Kepler, Newton, Bradley, Lagrange and William Herschel. The numerous illustrations throughout the work are very good and the reproductions from original photographs are first class. The book is one of great interest to the Astronomer as well as to the ordinary student and amateur. B. F. F.

Mathematical Essays and Recreations. By Herman Schubert, Professor of Mathematics in the Johanneum, Hamburg, Germany. Translated from the German by Thomas J. McCormack. Red Cloth, 148 pages. Price, 75 cents. Chicago : The Open Court Publishing Co.

The following subjects are discussed in a popular though scientific manner: (1) "The Definition and Notion of Number," (2) "Monism in Arithmetic," (3) "On the Nature of Mathematical Knowledge," (4) "Magic Squares," (5) "The Fourth Dimension," (6) "The History of the Squaring of the Circle."

The first three articles of the book are concerned with the construction of arithmetic as a monistic science. Number is defined as the result of counting. Fractional numbers, complex numbers, negative numbers, irrational numbers, imaginary numbers are all extensions of primitive results, made according to what Hankel calls "the principle of permanence," and Schubert the "principle of no exception." Arithmetic thus takes the general shape of a system of logical forms having consistency and coherency among themeelves. Professor Schubert being one of the most successful teachers in Germany, his sketch on monistic arithmetic will be found exceedingly interesting and suggestive. Any other theory of the number concept seems to me absolutely untenable.

The article on "Fourth Dimension" is a clear and easily understood exposition of the apparently mysterious subject. Professor Schubert clearly shows what is meant by dimensions as used in science and what is the legitimate function of a fourth dimension in mathematics. This article alone is worth the price of the book. The "History of the Squaring of the Circle" is full of interest from first to last, and is quite complete.

B. F. F.

The Elements of Physics. A Text-book for High Schools and Academies. By Alfred Payson Gage, Ph. D., Author of Principles of Physics, Introduction to Physics, etc. Revised Edition. 12mo. Half Leather, 381 pages. Introduction price, $1.12. New York and Chicago: Ginn & Co.

In bringing out the revision of this book, all the excellencies of the original work have been retained while many improvements in method of presentation have been introduced. Recent advances in the industrial applications of physical principles have received due attention. A large number of problems and practical exercises are furnished throughout the book.

B. F. F.

A Text-book of General Astronomy for Colleges and Scientific Schools. By Charles A. Young, Ph. D., LL. L., Professor of Astronomy in Princeton University. Revised Edition. 8vo. Half Leather, 630 pages. Price, $2.50. Boston, U. S. A., and London: Ginn & Co.

In bringing out the revised edition of this book, the best book on astronomy has been improved. Our acquaintance with this work was made by its use for several years in the class-room, and we have found it very satisfactory in every particular. The revised edition embodies the new and important results which have been obtained during the last ten years, and thereby the continued popularity of the work is insured for sometime in the future.

B. F. F.

A Text-book of Physics. By G. A. Wentworth, Author of a Series of Text-books in Mathematics, formerly Professor of Mathematics in Phillips Exeter Academy, and G. A. Hill, Author of Geometry for Beginners, formerly Assistant Professor of Physics in Harvard University, 12mo. Half Leather, 440 pages. Price for Introduction, $1.15. Boston and Chicago: Ginn & Co.

This work aims to give a rational explanation of the more important physical phenomena, and to prepare the way for further investigation and study of physical sciences. The book has many points of excellence to commend it to public favor. B. F. F.

Errata.

Page 42, line 9, for "$\angle CEM$" read $\angle CFM$.

line 12, for "$A-60^\circ$" read $B-60^\circ$.

line 22, for "angle FCL" read angle FCM.

line 23, for "angle EMC" read angle FMC.

line 24, for "$x+A+C-60^\circ$" read $x+B+C-60^\circ$.

line 24, for "$A+x+y$" read $B+x+y$.

Vol. V., No. 10, page 231, line 7, of solution, for "$\triangle ABC = \triangle CPB$" read $\triangle ABC + \triangle CPB$.

Same, line 8, for "$\frac{r_1 b}{2} = \frac{r_1 c}{2}$" read $\frac{r_1 b}{2} + \frac{r_1 c}{2}$.

Same, line 11, for "$= \frac{r_2 a}{2}$" read $+ \frac{r_2 a}{2}$.

SOPHUS LIE.

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BIOGRAPHY.

SOPHUS LIE.

BY GEORGE BRUCE HALSTED.

ON the eighteenth of February, 1899, the greatest mathematician in the world, Sophus Lie, died at Christiania in Norway.

He was essentially a geometer, though applying his splendid powers of space creation to questions of analysis. From Lie comes the idea that every system of geometry is characterized by its group.

In ordinary geometry a surface is a locus of points ; in Lie's *Kugel-geometrie* it is the aggregate of spheres touching this surface. By a simple correlation of this sphere-geometry with Pluecker's line-geometry, Lie reached results as unexpected as elegant. The transition from this line-geometry to this sphere-geometry was an example of contact-transformations.

Now contact-transformations find application in the theory of partial differential equations, whereby this theory is vastly clarified. Old problems were settled as sweepingly as new problems were created and solved. Again, with his *Theorie der Transformationsgruppen*, Lie changed the very face and fashion of modern mathematics.

A magnificent application of his theory of continuous groups is to the general problem of non-Euclidean geometry as formulated by Helmholtz. To this was awarded the great Lobachevski Prize. Not even this award could sufficient-

ly emphasize the epoch-making importance of Lie's work in the evolution of geometry.

Moreover, the foundations of all philosophy are involved. To know the non-Euclidean geometry involves abandonment of the position that axioms as to their concrete content are necessities of the inner intuition ; likewise abandonment of the position that axioms are derivable from experience alone.

Lie said that in the whole of modern mathematics the weightiest part is the theory of differential equations, and, true to this conviction, it has always been his aim to deepen and advance this theory.

Now it may justly be maintained that in his theory of transformation groups Lie has himself created the most important of the newer departments of mathematics.

By the introduction of his concept of continuous groups of transformations he put the isolated integration theories of former mathematicians upon a common basis.

The masterly reach of Lie's genius is illustrated by his encompassment of the fundamentally important theory of differential invariants associated with the English names Cayley, Cockle, Sylvester, Forsyth.

Thirteen years ago Sylvester announced his conception of 'Reciprocants,' a body of differential invariants not for a group, but for a mere interchange of variables. A number of Englishmen thereupon took up investigations about orthogonal, linear and projective groups, groups in whose transformations interchanges of variables occur as particular cases, and whose differential invariants are consequently classes of reciprocants, and of the analogues of reciprocants, when more variables than two are considered.

Now all these investigations were long subsequent to Lie's consideration of the groups in question as leading cases of a general conception. Thus they were merely secondary investigations !

Again the theory of complex numbers appears as a part of the great 'Theorie der Transformationsgruppen.' Indeed, this continent of 'transformations' opened up and penetrated with such giant steps by Lie represents the most remarkable advance which mathematics in all its entirety has made in this latter part of the century.

Sophus Lie it was who made prominent the importance of the notion of group, and gave the present form to the theory of continuous groups. This idea, like a brilliant dye, has now so permeated the whole fabric of mathematics that Poincaré actually finds that in Euclid 'the idea of the group was potentially preexistent,' and that he had 'some obscure instinct for it, without reaching a distinct notion of it.' Thus the last shall be first and the first last.

In personal character Lie was our ideal of a genius, approachable, outspoken, unconventional, yet at times fierce, intractable.

His work is cut short ; his influence, his fame, will broaden, will tower from day to day.

SOPHUS LIE.*

BY PROFESSOR GASTON DARBOUX.

Sophus Lie was born on the 17th of December, 1842, at Nordfjordeid (near Florö) where his father, John Herman Lie, was pastor. The studies of his childhood and youth did not reveal in him that exceptional aptitude for mathematics which is signalized so early in the lives of the great geometers: Gauss, Abel, and many others. Even on leaving the University of Christiania in 1865, he still hesitated between philology and mathematics. It was the works of Plücker on modern geometry which first made him fully conscious of his mathematical abilities and awakened within him an ardent desire to consecrate himself to mathematical research. Surmounting all difficulties and working with indomitable energy he published his first work in 1869, and we can say that from 1870 on he was in possession of the ideas which were to direct his whole career.

At this time I frequently had the pleasure of meeting and conversing with him in Paris where he had come with his friend F. Klein. A course of lectures by Sylow revealed to Lie all the importance of the theory of substitution groups; the two friends studied this theory in the great treatise of our colleague Jordan; they saw fully the essential rôle which it would be called upon to play in all the branches of mathematics to which it had then not been applied. They have both had the good fortune to contribute by their works to impressing upon mathematical studies the direction which appeared to them to be the best.

A short note of Lie "Sur une transformation géométrique," presented to our Academy in October, 1870, contains an extremely original discovery. Nothing resembles a sphere less than a straight line and yet, by using the ideas of Plücker, Lie found a singular transformation which makes a sphere correspond to a straight line, and which consequently makes possible the derivation of a theorem relative to an ensemble of spheres from every theorem relative to an aggregate of straight lines, and vice versa. It is true that if the lines are real, the corresponding spheres are imaginary. But such difficulties are not sufficient to deter geometers. In this curious method of transformation, each property relative to asymptotic lines of a surface is transformed into a property relative to lines of curvature. The name of Lie will remain attached to these concealed relations which connect the two essential and fundamental elements of geometric investigation, the straight line and sphere. He has developed them in detail in a memoir full of new ideas which appeared in 1872 in the *Mathematische Annalen*.

The works following this brilliant beginning fully confirmed all the hopes to which it gave birth. Since the year 1872 Lie has put forth a series of memoirs upon the most difficult and most advanced parts of the integral calculus. He

*From the Bulletin of the American Mathematical Society. Translated by Edgar Odell Lovett from *Comptes rendus*.

commences by a profound study of the works of Jacobi on the partial differential equations of the first order and at first coöperates with Mayer in perfecting this theory in an essential point. Then, by continuing the study of this beautiful subject, he is led to construct progressively that masterful theory of continuous transformation groups which constitutes his most important work and in which, at least at the start, he was aided by no one. The detailed analysis of this vast theory would require too much space here. It is proper, however, to point out particularly two elements wholly essential to these researches: first, the use of contact transformations which throws such a vivid and unexpected light upon the most difficult and obsure parts of the theories relative to the integration of partial differential equations; second, the use of infinitesimal transformations. The introduction of these transformations is due entirely to Lie; their use, like that of Lagrange's variation, naturally greatly extends both the notion of differential and the applications of the infinitesimal calculus.

The construction of so extended a theory did not satisfy Lie's activity. In order to show its importance he has applied it to a great number of particular subjects, and each time he has had the good fortune of meeting with new and elegant properties. I find my preference in the researches which he has published since 1876 on minimal surfaces. The theory of these surfaces, the most attractive perhaps that presents itself in geometry, still awaits, and may await a long time, the complete solution of the first problem to be proposed in it, namely, the determination of a minimal surface passing through a given contour. But, in return, it has been enriched by a great number of interesting propositions due to a multitude of geometers. In 1866 Weierstrass made known a very precise and simple system of formulæ which has called forth a whole series of new studies on these surfaces. In his works Lie returns simply to the formulæ of Monge; he gives their geometric interpretation and shows how their use can lead to the most satisfactory theory of minimal surfaces. He makes known methods which permit of determining all algebraic minimal surfaces of given class and order. Finally, he studies the following problem: to determine all algebraic minimal surfaces inscribed in a given algebraical developable surface. He gives the complete solution for the case where only one of these surfaces inscribed in the developable is known.

Of great interest also are the researches which we owe to him on the surfaces of constant curvature, in the study of which he makes use of a theorem of Bianchi on geodesic lines and circles, likewise those on surfaces of translation, on the surfaces of Weingarten, on the equations of the second order having two independent variables, et cetera. I should reproach myself for forgetting, even in so rapid a résumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge.

These extensive works quickly attracted to the great geometer the attention of all those who cultivate science or are interested in its progress. In 1877 a new chair of mathematics was created for him at the University of Christiania,

and the foundation of a Norwegian review enabled him to pursue his work and publish it in full. In 1886, he accepted the honor of a call to the University of Leipzig; he taught in this university with the rank of ordinary professor from 1886 to 1898. To this period of his life is to be referred the publication of his didactic works, in which he has coördinated all his researches. Six months ago he returned to his native land to assume at Christiania the chair which had been especially reserved for him by the Norwegian parliament, with the exceptional salary of ten thousand crowns. Unfortunately, excess of work had exhausted his strength and he died of cerebal anæmia at the age of fifty-six years.

Nowhere is his loss felt more keenly than in our country, where he had so many friends. True, in 1870 a misadventure befell him, whose consequences I was instrumental in averting. Surprised at Paris by the declaration of war, he took refuge at Fontainebleau. Occupied incessantly by the ideas fermenting in his brain, he would go every day into the forest, loitering in places most remote from the beaten path, taking notes and drawing figures. It took little at this time to awaken suspicion. Arrested and imprisioned at Fontainebleau, under conditions otherwise very comfortable, he called for the aid of Chasles, Bertrand, and others; I made the trip to Fontainebleau and had no trouble in convincing the procureur impérial; all the notes which had been seized and in which figured complexes, orthogonal systems, and names of geometers, bore in no way upon the national defenses. Lie was released; his high and generous spirit bore no grudge against our country. Not only did he return voluntarily to visit it but he received with great kindness French students, scholars of our École Normale who would go to Leipzig to follow his lectures. It is to the École Normale that he dedicated his great work on the theory of transformation groups. A number of our thesis at the Sorbonne have been inspired by his teaching and dedicated to him.

The admirable works of Sophus Lie enjoy the distinction, to-day quite rare, of commanding the common admiration of geometers as well as analysts. He has discovered fundamental propositions which will preserve his name from oblivion, he has created methods and theories which, for a long time to come, will exercise their fruitful influence on the development of mathematics. The land where he was born and which has known how to honor him can place with pride the name of Lie beside that of Abel, of whom he was a worthy rival and whose approaching centenary he would have been so happy in celebrating.

ON THE SIMPLE GROUPS WHICH CAN BE REPRESENTED AS SUBSTITUTION GROUPS THAT CONTAIN CYCLICAL SUBSTITUTIONS OF A PRIME DEGREE.

By DR. G. A. MILLER.

It is known that cyclical substitutions of a prime degree (p) cannot occur in any primitive group (if it is not alternating or symmetric) unless the degree of the primitive group (G) is one of the following three numbers, p, $p+1$, $p+2$. When G is of degree p all its substitutions of order p generate a simple group which is a selfconjugate subgroup of G (if it does not coincide with G) and corresponds to identity of a cyclical quotient group of G, the order of this quotient group being a divisor of $p-1$.* We proceed to consider the cases when G contains substitutions of order p and is either of degree $p+1$ or of degree $p+2$.

If G is of degree $p+2$ and is not generated by its substitutions of order p it must contain a selfconjugate subgroup (G') which is generated by these substitutions, G' is at least triply transitive since it contains substitutions of order p.† We proceed to prove that it is a simple group when p exceeds 2. If it were compound each of its at least doubly transitive maximal subgroups of degree $p+1$ would also be compound, since the substitutions of such a subgroup (G_1) which would belong to the given selfconjugate subgroup of G, would form a selfconjugate subgroup of G_1.

A maximal subgroup (G_2) of degree p that is contained in G_1 must be transitive. Since every selfconjugate subgroup of a transitive group of degree p must contain substitutions of order p, G_2 could have only identity in common with the given self-conjugate subgroup of G. Hence the order of this selfconjugate subgroup would be $(p+1)(p+2)$, and the corresponding quotient group would be simply isomorphic to G_2. To a subgroup of order p that is contained in G_2 there would correspond a subgroup of G, whose order would be $p(p+1)(p+2)$. Since this is evidently impossible we have the important

THEOREM. *All the substitutions of order p (p being any odd prime number) that are contained in any primitive group of degree p or of degree $p+2$ generate a simple group. If this simple group does not coincide with the entire group it is selfconjugate and the corresponding quotient group is cyclical and has for its order a divisor of $p-1$.‡ This simple group cannot be selfconjugate subgroup of more than one group of its own degree and of a given order.*

We shall now consider the groups of degree $p+1$, p being any odd prime number, that contain substitutions of order p. Such groups are at least doubly transitive and their substitutions of order p generate a doubly transitive selfconjugate subgroup if they do not generate the entire group. Let H be such a sub-

*Cf. *Bulletin of the American Mathematical Society*, Vol. 4, 1898, page 140.

†Jordan, *Journal de Mathematiques*, Vol. 16, 1871.

‡From this theorem it follows directly that the primitive group of degree 9 and of order 504 is simple.

group. We see as in the preceding paragraph that H cannot contain any selfconjugate subgroup unless this subgroup is a regular group of order $p+1$. As this regular subgroup has to contain p subgroups that are conjugate in H it cannot involve any substitution whose order exceeds 2. Hence we have the

THEOREM. *If $p+1$ is not a power of 2 then the substitutions of order p (p being any odd prime number) that are contained in a group of degree $p+1$ generate a simple group. If this simple group does not coincide with the entire group it is selfconjugate and the corresponding quotient group is a cyclical and its order is a divisor of $p-1$.** *If $p+1$ is a power of 2, the group generated by the substitutions of order p that are contained in a group of degree $p+1$ cannot contain any selfconjugate subgroup except perhaps the regular group of order $p+1$ which contains no substitution whose order exceeds 2.*

Cornell University, April 3, 1899.

*From this theorem it follows directly that the three primitive groups of degree 12 and orders 660, 7920, and 95040, respectively, are simple. The last of these three groups is the well known five-fold transitive group of Mathien.

ON SYMMETRIC FUNCTIONS.

By E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from March Number.]

B. FUNDAMENTAL RELATIONS FOR SYMMETRIC FUNCTIONS.

1. FUNDAMENTAL RELATIONS BETWEEN COEFFICIENTS.

(1). *Derivation of the relations.*

In A, 4, (3) we have already obtained one of the relations, viz :

$$\binom{0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n}}{b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_m{}^{\kappa_n}}=(-1)^{mn}\binom{(m-\kappa_1)(m-\kappa_2)\ldots\ldots(m-\kappa_n)}{a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0}.$$

If in the equations $b_0x^n+b_1x^{n-1}+\ldots\ldots+b_n=0$, and $a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m=0$ (cf. loc. cit.), we substitute $x=1/y$, b_r becomes b_{n-r}, a_r becomes a_{m-r}, and $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}$ becomes

$$b_n{}^m\,\Sigma\frac{1}{\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}}=\frac{b_n{}^m}{(\beta_1\beta_2\ldots\ldots\beta_n)^m}\Sigma\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}$$

$$=(-1)^{mn}b_0{}^m\Sigma\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}.$$

Similarly $a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0$ becomes $(-1)^{mn}a_0{}^n\Sigma(\alpha\lambda_n)^n(\alpha\lambda_{n-1})^{n-1}\ldots\ldots(\alpha\lambda_0)^0$. We have therefore,

$$\binom{0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n}}{b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}}=(-1)^{mn}\binom{0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots\ldots n^{\lambda_0}}{b_0{}^m\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}} \text{ and}$$

$$\binom{(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n)}{a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots(\alpha\lambda_n)^0}=(-1)^{mn}\binom{\varkappa_1\varkappa_2\ldots\ldots\ldots\ldots\ldots\varkappa_n}{a_0{}^n(\alpha\lambda_n)^n(\alpha\lambda_{n-1})^{n-1}\ldots(\alpha\lambda_0)^0}$$

The last relations correspond to reciprocal terms in the resultant theory. [Terms of the resultant like $(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ and $(a_{m-r_\mu})^{p_\mu}(a_{m-r_{\mu-1}})^{x_{\mu-1}}\ldots(a_{m-r_1})^{p_1}(b_{n-s_\nu})^{q_\nu}(b_{n-s_{\nu-1}})^{q_{\nu-1}}\ldots(b_{n-s_1})^{q_1}$ are called reciprocal terms.] Since one member in each is equal numerically to the terms of the relation already obtained, the four terms are numerically equal. Taken together we shall call them the fundamental relations for the coefficients of symmetric functions.

(2). *Other notation for the fundamental relation.*

We will now re-write the relations given in (1), replacing such expressions as $b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\beta_n{}^{\kappa_n}$ and $a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots(\alpha\lambda_n)^0$ by $0^m\varkappa_1\varkappa_2\ldots\varkappa_n$ and $0^n n^{\lambda_0}(n-1)^{\lambda_1}\ldots0^{\lambda_n}$, respectively, $(n-r)^{\lambda_r}$, *e. g.*, signifying, as before $(\alpha\lambda_r)^{n-r}$ signified, that λ_r roots have the exponent $(n-r)$. We may then write the fundamental relations :

$$\binom{0^{\lambda_0}1^{\lambda_1}\ldots\ldots n^{\lambda_n}}{0^m\varkappa_1\varkappa_2\ldots\varkappa_m}=\binom{\varkappa_1\varkappa_2\ldots\ldots\varkappa_n}{0^n0^{\lambda_0}1^{\lambda_1}\ldots n^{\lambda_n}}=(-1)^{mn}\binom{0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots n^{\lambda_0}}{0^m(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n)}$$

$$=(-1)^{mn}\binom{(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n)}{0^n0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots n^{\lambda_0}}.^*$$

2. Fundamental and Necessary Conditions of Coefficients.

We will next consider some relations of conditions of the coefficients themselves.

(1). *First condition.*

Since $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}=b_0{}^m(\beta_1\beta_2\ldots\ldots\beta_n)^{\kappa_n}\Sigma\beta_1{}^{\kappa_1-\kappa_n}\beta_2{}^{\kappa_2-\kappa_n}\ldots\ldots(\beta_{n-1})^{\kappa_{n-1}-\kappa_n}=(-1)^{n\kappa_n}b_0{}^{m-\kappa_n}b_n{}^{\kappa_n}\Sigma\beta_1{}^{\kappa_1-\kappa_n}\beta_2{}^{\kappa_2-\kappa_n}\ldots(\beta_{n-1})^{\kappa_{n-1}-\kappa_n}$, supposing $\varkappa_1,\varkappa_2,\ldots\varkappa_n$ to be in order of descending magnitude, it follows that, at least, either $\lambda_n>0$, (and $\lambda_n\geq\varkappa_n$), or else $\lambda_n=0$, and $\varkappa_n=0$. This corresponds to the condition in the resultant $R_{m,n}$, that, at least, one of the two factors b_n or a_m must be present in every term.

(2). *Second condition.*

Again, since by 1, (2),

*It might seem as if the distinction of a and b is lost by this notation. A little reflection shows that this distinction does not need to be kept. In fact the theory is clearer without it.

$$\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\ldots\ldots n^{\kappa_n}\\0^m\varkappa_1\varkappa_2\;\ldots\;\varkappa_n\end{pmatrix}=(-1)^{mn}\begin{pmatrix}0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0}\\0^m(m-\varkappa_1)(m-\varkappa_2)\;\ldots\;(m-\varkappa_n)\end{pmatrix}$$

it follows by the previous proof, that, at least, either $\lambda_0>0$, (and $\lambda_0\geqq m-\varkappa_1$), or else $\lambda_0=0$, $m-\varkappa_1=0$, *i. e.*, $\varkappa_1=m$. This corresponds to the condition in the resultant $R_{m,n}$, that, at least of the two factors b_0 and a_0, one must be present in every term.

When these conditions are not satisfied, the coefficient is equal to zero. It will be shown later that if $\lambda_n=\lambda_{n-1}=\ldots=\lambda_r=0$. then $\varkappa_n=\varkappa_{n-1}=\ldots=0$, and if $\lambda_0=\lambda_1=\ldots=\lambda_r=0$, then $m-\varkappa_1=m-\varkappa_2=\ldots=m-\varkappa_r=0$, *i. e.*, $\varkappa_1=\varkappa_r=\ldots=\varkappa_r=m$; otherwise the coefficient is zero (D, 3 and 4).

3. Fundamental Equations of Condition Connecting the Subscripts and Exponents.

From the theorems of order and weight of symmetric functions, and from the relations between the four coefficients given in 1, (2), we have the following equations of condition connecting the subscripts and exponents:

(1). $\lambda_0+\lambda_1=\ldots\lambda_n=m$,

(2). $\lambda_1+2\lambda_2+\ldots n\lambda_n=\varkappa_1+\varkappa_2+\ldots\varkappa_n$,

(3). $\lambda_{n-1}+2\lambda_{n-2}+\ldots n\lambda_0=mn-(\varkappa_1+\varkappa_2+\ldots\varkappa_n)$, and by adding mn to both sides of (2) that equation may be written,

(4). $\lambda_1+2\lambda_2+\ldots n\lambda_n+mn-(\varkappa_1+\varkappa_2+\ldots\varkappa_n)=mn$, the same equations which we should have in the theory of the resultant, excepting (3) which can be derived from (1) by multiplying that equation by n, and then substituting the value of $\varkappa_1+\varkappa_2+\ldots\varkappa_n$ from (2).

C. NORMAL FORMS AND REDUCIBLE FORMS.

As in the theory of the resultant so also here we have normal forms and reducible forms. A normal form will be characterized by having $\lambda_0>0$, $\lambda_n>0$, $\varkappa_1=m$, $\varkappa_n=0$. All other forms, the completely reducible forms excepted, may be reduced to such as are normal forms having a lower m or n. As in the rerultant theory, so here there are four kinds of reduction, with the four corresponding kinds of derivation. In the next divisions we will briefly treat of these.

D. REDUCTION.

1. Reduction in the Case where $\varkappa_1=m$.

In this case λ_0 must be greater than zero [cf. B, 2, (2)]. It is evident that we may divide by the factor b_0 in the given term as well as in $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\beta_n{}^{\kappa_n}$ without affecting the value of the coefficient of the term, and that therefore,

$$\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\;\ldots\;n^{\lambda_n}\\0^m\varkappa_1\varkappa_2\;\ldots\ldots\varkappa_n\end{pmatrix}=\begin{pmatrix}0^{\lambda_0-1}\,1^{\lambda_1}\ldots\ldots n^{\lambda_n}\\0^{m-1}\varkappa_1\varkappa_2\;\ldots\ldots\varkappa_n\end{pmatrix},$$

and this reduction may be continued until the exponent of b_0 below is equal to $\varkappa_1$. We then have

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0-\lambda} 1^{\lambda_1} \ldots . n^{\lambda_n} \\ 0^{\kappa_1} \varkappa_1 \varkappa_2 \ldots . \varkappa_n \end{pmatrix},$$

where $m-\lambda=\varkappa$, $\lambda_0-\lambda\geqq 0$.

This reduction corresponds to the third kind of reduction in the theory of the resultant.

2. Reduction in the Case where $\varkappa_n \neq 0$.

Here we must have $\lambda_n>0$. [Cf. B, 2, (1)], and the work already done in B, 2, shows that

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots . n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{n\kappa_n} \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n-\kappa_n} \\ 0^{m-\kappa_n} (\varkappa_1-\varkappa_n)(\varkappa_2-\varkappa_n) \ldots (\varkappa_{n-1}-\varkappa_n) \end{pmatrix}$$

or we may prove it otherwise as follows: By B, 1, (2),

$$=\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots . n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots . \varkappa_n \end{pmatrix} = (-1)^{mn} \begin{pmatrix} 0^{\lambda_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m (m-\varkappa_1)(m-\varkappa_2) \ldots . (m-\varkappa_n) \end{pmatrix}.$$

By 1, the right member of this can be reduced. We get

$$\begin{pmatrix} 0^{\lambda_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m (m-\varkappa_n)(m-\varkappa_{n-1}) \ldots . (m-\varkappa_1) \end{pmatrix} = \begin{pmatrix} 0^{\lambda_n-\kappa_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^{m-\kappa_n} (m-\varkappa_n)(m-\varkappa_{n-1}) \ldots . (m-\varkappa_1) \end{pmatrix}.$$

Again by B, 1, (2), $\begin{pmatrix} 0^{\lambda_n-\kappa_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^{m-\kappa_n} (m-\varkappa_n)(m-\varkappa_{n-1}) \ldots (m-\varkappa_1) \end{pmatrix}$

$$=(-1)^{(m-\kappa_n)n} \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n-\kappa_n} \\ 0^{m-\kappa_n} (\varkappa_1-\varkappa_n)(\varkappa_2-\varkappa_n) \ldots (\varkappa_{n-1}-\varkappa_n) \end{pmatrix}.$$

Substituting, we get

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{n\kappa_n} \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n-\kappa_n} \\ 0^{m-\kappa_n} (\varkappa_1-\varkappa_n)(\varkappa_2-\varkappa_n) \ldots . (\varkappa_{n-1}-\varkappa_n) \end{pmatrix}$$

as before. This corresponds to the first kind of reduction in the resultant theory.

3. Reduction in the Case where $\lambda_n=0$.

We must have $\kappa_n=0$ in this case. By B, 1, (2) we have

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots . \varkappa_{n-1} 0 \end{pmatrix} = \begin{pmatrix} 0\ \varkappa_{n-1}\ \varkappa_{n-2} \ldots \varkappa_1 \\ 0^n\ 0^{\lambda_0}\ 1^{\lambda_1} \ldots\ldots n^0 \end{pmatrix}.$$

By D, 1, the right member can be reduced, and

$$\begin{pmatrix} 0\ \varkappa_{n-1}\ \varkappa_{n-2} \ldots\ldots\ldots\ldots \varkappa_1 \\ 0^n\ 0^{\lambda_0}\ 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}}\ n^0 \end{pmatrix} = \begin{pmatrix} 0\ \varkappa_{n-1}\ \varkappa_{n-2} \ldots\ldots\ldots\ldots \varkappa_1 \\ 0^n\ 0^{\lambda_0}\ 1^{\lambda_1} \ldots . (n-1)^{\lambda_{n-1}} \end{pmatrix}$$

$$=\begin{pmatrix} \varkappa_{n-1}\ \varkappa_{n-2} \ldots\ldots \ldots\ldots \varkappa_1 \\ 0^{n-1}\, 0^{\lambda_0}\, 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}} \end{pmatrix}.$$

Again by B, 1, (2),

$$\begin{pmatrix} \varkappa_{n-1}\ \varkappa_{n-2} \ldots\ldots\ldots \varkappa_1 \\ 0^{n-1} 0^{\lambda_0}\, 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}} \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_{n-1} \end{pmatrix},$$

and by substituting,

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}} n^0 \\ 0^m \varkappa_1 \varkappa_2 \ \ldots\ldots \varkappa_{n-1}\ 0 \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_{n-1} \end{pmatrix}.$$

In a similar way it will follow that if $\lambda_n=\lambda_{n-1}=\ldots=\lambda_r=0$, $\varkappa_n=\varkappa_{n-1}=\ldots=\varkappa_r=0$, and

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots r^0 (r+1)^0 \ldots n^0 \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_{r-1} 0^{n-r+1} \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots (r-1)^{\lambda_{r-1}} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_{r-1} \end{pmatrix}.$$

The meaning of this last formula is that a function of the roots of an equation of the nth degree, which involves only $(r-1)$ roots at a time, has the same coefficient of the term involving $b_0^{\lambda_0} b_1^{\lambda_1} \ldots (b_{r-1})^{\lambda_{r-1}}$, as the same function of the roots of an equation of the $(r-1)$st degree which involves all the roots ; and the reduction corresponds to the reduction of the second kind in the resultant theory.

4. Reduction in the Case where $\lambda_0=0$.

We must have here $\varkappa_1=m$. By B, 1, (2), and D, 3,

$$\begin{pmatrix} 1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n} \\ 0^{\kappa_1} \varkappa_1 \varkappa_2 \ldots \varkappa_n \end{pmatrix} = (-1)^{\kappa_1 n} \begin{pmatrix} 0^{\lambda_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots (n-1)^{\lambda_1} \\ 0^{\kappa_1} (\varkappa_1-\varkappa_n)(\varkappa_1-\varkappa_{n-1}) \ldots (\varkappa_1-\varkappa_2) \end{pmatrix}.$$

Again by B, 1, (2),

$$\begin{pmatrix} 0^{\lambda_n}\, 1^{\lambda_{n-1}} \ldots\ldots\ldots (n-1)^{\lambda_1} \\ 0^{\kappa_1} (\varkappa_1-\varkappa_n)(\varkappa_1-\varkappa_{n-1}) \ldots (\varkappa_1-\varkappa_2) \end{pmatrix} = (-1)^{\kappa_1 (n-1)} \begin{pmatrix} 0^{\lambda_1}\, 1^{\lambda_2} \ldots (n-1)^{\lambda_n} \\ 0^{\kappa_1} \varkappa_2 \varkappa_3 \ldots\ldots \varkappa_n \end{pmatrix}.$$

Therefore $\begin{pmatrix} 1^{\lambda_1} 2^{\lambda_2} \ldots\ldots n^{\lambda_n} \\ 0^{\kappa_1} \varkappa_1 \varkappa_2 \ldots \varkappa_n \end{pmatrix} = (-1)^{\kappa_1} \begin{pmatrix} 0^{\lambda_1}\, 1^{\lambda_2} \ldots (n-1)^{\lambda_n} \\ 0^{\kappa_1} \varkappa_2 \varkappa_3 \ldots \ldots \varkappa_n \end{pmatrix}.$

If $\lambda_0=\lambda_1=\ldots=\lambda_{r-1}=0$.

$$\begin{pmatrix} r^{\lambda_r} (r+1)^{\lambda_{r+1}} \ldots n^{\lambda_r} \\ 0^m \varkappa_1\ \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{mn} \begin{pmatrix} 0^{\lambda_n}\, 1^{\lambda_{n-1}} \ldots\ldots (n-r)^{\lambda_r} \\ 0^m (m-\varkappa_1)(m-\varkappa_2) \ldots (m-\varkappa_n) \end{pmatrix},$$

and the right member of this is either zero, or else

$$m-\varkappa_1=m-\varkappa_2=\ldots=m-\varkappa_r=0,\ \varkappa_1=\varkappa_2=\ldots=\varkappa_r=m,$$

and the reduction of this section gives

$$\begin{pmatrix} r^{\lambda_r}(r+1)^{\lambda_{r+1}} \ldots n^{\lambda_n} \\ 0^m \varkappa_1\ \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{\kappa_1 r} \begin{pmatrix} 0^{\lambda_r}\, 1^{\lambda_{r+1}} \ldots (n-r)^{\lambda_n} \\ 0^{\kappa_1} \varkappa_{r+1} \varkappa_{r+2} \ldots\ldots \varkappa_n \end{pmatrix}.$$

This kind of reduction corresponds to reduction of the fourth kind in the resultant theory.

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

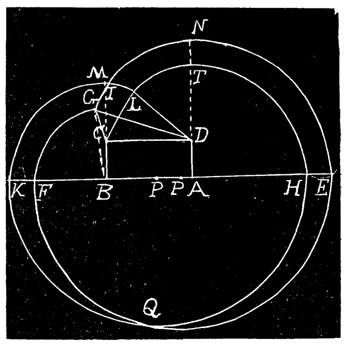
ARITHMETIC.

107. Proposed by REBER V. ALLEN, Hooker Station, Ohio.

A barn, $ABCD$, length $AB=b$ feet, width $AD=a$ feet, standing in an open field, has a horse tethered to a point, P, in the side, AB, distance $AP=c$ feet, with a rope R feet long. Over what area can the horse graze ?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

We might consider six cases of which only the fifth and sixth are represented in the figure.



Case I. In every case $c<b-c$. $R>c$ and $<b-c$, also $<a+c$.

Area$=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2=\frac{1}{4}\pi(3R^2-2Rc+c^2)$.

Case II. $R>c$, $R>b-c$, $R<a+c$, $R<a+b-c$.

Area$=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-b+c)^2=\frac{1}{4}\pi(4R^2+2c^2+b^2-2Rb-2bc)$.

Case III. $R>c+a$, $R<a+b-c$.

Area$=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-c-a)^2+\frac{1}{4}\pi(R-b+c)^2=\frac{1}{4}\pi(5R^2-2Rc+3c^2+a^2-2Ra+2ac+b^2-2Rb-2bc)$.

Case IV. $R>c+a$, $R>a+b-c$, $R<a+b$.

Area$=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-c-a)^2+\frac{1}{4}\pi(R-b+c)^2+\frac{1}{4}\pi(R-a-b+c)^2=\frac{1}{2}\pi(3R^2+2c^2+a^2-Ra+b^2-2Rb-2bc+ab)$.

Case V. $R>a+b$; intersection at L between AD, BC produced.

Area$=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-b+c)^2+$sector $DTL+$sector $CIL+$triangle DLC.

$DL=R-a-c$, $DC=b$, $CL=(R-a-b+c)$.

$$\therefore\ \angle LDC=\theta=\cos^{-1}\left(\frac{b^2+(R-a-c)^2-(R-a-b+c)^2}{2b(R-a-c)}\right)$$

$$\angle DCL=\varphi=\cos^{-1}\left(\frac{b^2+(R-a-b+c)^2-(R-a-c)^2}{2b(R-a-b+c)}\right).$$

$\therefore$ Area$=\frac{1}{4}\pi(4R^2+2c^2+b^2-2Rb-2bc)+\frac{1}{2}(R-a-c)^2(\frac{1}{2}\pi-\theta)+\frac{1}{2}(R-a-b+c)^2(\frac{1}{2}\pi-\varphi)+\frac{1}{2}b(R-a-c)\sin\theta$.

Case VI. $R>a+b$; intersection in angle KBM.

Area$=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+$sector $NDG+$sector $FBG+$triangle $CDG+$triangle CBG.

$DG=(R-a-c)$, $DC=b$, $BC=a$, $BG=(R-b+c)$, $DB=\sqrt{(a^2+b^2)}$.

$$\therefore \angle DBG=\beta=\cos^{-1}\left(\frac{a^2+b^2+(R-b+c)^2-(R-a-c)^2}{2(R-b+c)\sqrt{(a^2+b^2)}}\right)$$

$$\angle GDB=\gamma=\cos^{-1}\left(\frac{a^2+b^2+(R-a-c)^2-(R-b+c)^2}{2\sqrt{(a^2+b^2)}(R-a-c)}\right)$$

$$\angle CBD=\beta'=\cos^{-1}\left(\frac{a}{\sqrt{(a^2+b^2)}}\right)=\tan^{-1}\left(\frac{b}{a}\right)$$

$\angle CDB=\frac{1}{2}\pi-\beta'$, $\angle GDC=\gamma+\beta'-\frac{1}{2}\pi$, $\angle CBG=\beta-\beta'$, triangle DCG $+$triangle $BCG=$triangle $DGB-$triangle DCB.

$\therefore$ Triangle $DCG+$triangle $BCG=\frac{1}{2}\sqrt{(a^2+b^2)}(R-a-c)\sin\gamma-\frac{1}{2}ab$.

$\therefore$ Area$=\frac{1}{4}\pi(3R^2-2Rc+c^2)+\frac{1}{2}(R-a-c)^2(\pi-\gamma-\beta')+\frac{1}{2}(R-b+c)^2(\beta-\beta')+\frac{1}{2}\sqrt{(a^2+b^2)}(R-a-c)\sin\gamma-\frac{1}{2}ab$.

II. Solution by C. C. BEBOUT, Professor of Mathematics in Elgin High School, Elgin, Ill.; CHARLES C. CROSS, Libertytown, Md., and ELMER SCHUYLER, High Bridge, N. J.

As shown by the figure, the area grazed over is made up of one semi-circle, two quadrants, two sectors, and a triangle. We must sum the areas of these.

Area of semi-circle $HPK=\frac{1}{2}\pi R^2$ ......(1).

Area of quadrant $HAK=\frac{1}{4}\pi(R-c)^2$ ......(2).

Area of quadrant $KBI=\frac{1}{4}\pi(R-b+c)^2$ ..... (3).

By using formula $A=\sqrt{[s(s-a)(s-b)(s-c)]}$ we get area of triangle DLC $=\sqrt{[c(R-a)(R-a-b)(b-c)]}$ ......(4).

To find the areas of the sectors we must find the angles TDL and ICL. These angles are respectively the complements of angles LDC and DCL, which may be found as angles of the triangle LCD, whose sides are known. To find these angles use the trigonometric formula:

$$\tan\tfrac{1}{2}A=+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Then $\tan\frac{1}{2}\angle LDC=\sqrt{\dfrac{c(R-a-b)}{(R-a)(b-c)}}$ and $\tan\frac{1}{2}\angle LCD=\sqrt{\dfrac{(R-a-b)(b-c)}{c(R-a)}}$

$$\therefore \angle LDT=(90^\circ-\angle LDC)=90^\circ-2\tan^{-1}\sqrt{\frac{c(R-a-b)}{(R-a)(b-c)}},$$

and $\angle LCI=(90^\circ-\angle LCD)=90^\circ-2\tan^{-1}\sqrt{\dfrac{(R-a-b)(b-c)}{c(R-a)}}$.

$$\therefore \text{ Area of sector } LDT=\frac{360^\circ}{90^\circ-2\tan^{-1}\sqrt{\dfrac{c(R-a-b)}{(R-a)(b-c)}}}[\pi(R-a-c)^2]\ldots\ldots (5),$$

$$\text{and area of sector } LCI=\frac{360^\circ}{90^\circ-2\tan^{-1}\sqrt{\dfrac{(R-a-b)(b-c)}{c(R-a)}}}[\pi(R-a-b+c)^2]\,..(6).$$

Summing (1), (2), (3), (4), (5), and (6), we get the area grazed over,

$$\tfrac{1}{4}\pi(4R^2+b^2+2c^2-2bR-2bc)+\sqrt{[c(R-a)(R-a-b)(b-c)]}$$

$$+\frac{360^\circ}{90^\circ-\tan^{-1}\sqrt{\dfrac{c(R-a-b)}{(R-a)(b-c)}}}[\pi(R-a-c)^2]$$

$$+\frac{360^\circ}{90^\circ-2\tan^{-1}\sqrt{\dfrac{(R-a-b)(b-c)}{c(R-a)}}}[\pi(R-a-b+c)^2].$$

Solutions of problem 106 were received from P. S. Berg and Elmer Schuyler, and of problem 105 from Sylvester Robins. These solutions came too late for credit in last issue.

NOTE ON THE CALCULATION OF INTEREST AND DISCOUNT.

BY JOSEPH V. COLLINS, PH. D., PROFESSOR OF MATHEMATICS, STATE NORMAL SCHOOL, STEVENS POINT, WIS.

In the January number of the MONTHLY, Hon. J. H. Drummond, after giving the answer to the problem, 'What are the proceeds of a note discounted at a bank for 10 years at 10 per cent.,' as *nothing*, says, "The method of calculating discount used by banks was invented to evade the usury laws. I have thought that the court which first sustained the method could not have been well versed in mathematical principles."

A curious thing and commentary on the preceding is that all methods of calculating interest and discount are open to objection. Custom requires that interest be paid at the end of each year or specified fraction of a year. When it is not paid until the end of a period of two or more years, the lender is defrauded of the interest on his interest. If all interest payments are made promptly, it is equivalent to paying compound interest. The business world recognizes that compound interest is the only fair kind. Thus tables of bond values and the like are always made on a compound interest basis. But simple interest is

the only kind which can be collected unless annual or compound is called for in the contract. The courts have decided, probably because compound interest piles up so rapidly, that borrowers shall not be compelled to pay it unless they have agreed to. It is evident that if simple interest is unfair to a lender, so likewise is its analogue in discount, viz., true discount. In bank discount the bank not only collects a certain rate of interest on the loan, but also *on its own pay*, which is theoretically an absurd proceeding.

In partial payments we find the same difficulties presenting themselves. If payments are made within a year, according to the United States rule interest is collected and set at interest before it is due. The old so-called Connecticut rule tried to avoid this unfairness, but in so doing became too complicated for general use. Business men see that the mercantile rule is the only one which is fair to both parties when the whole transactions falls within a year. But the mercantile rule works injustice if the period covers more than a year, since by it the lender gets only simple interest, whereas the United States rule gives him a form of compound interest. The United States rule works injustice to the borrower whenever he pays less than the interest. Thus it is evident that the element of time and certain practical considerations have a great deal to do towards determining the appropriate method of counting interest.

Looked at from a practical standpoint it is easy to see why the banker collects bank instead of true discount. True discount requires a long division after a preliminary calculation. Tables could not be made for computing true discount which would be at all convenient to use. The great bulk of the loans made by banks are for less than 3 mouths. The difference between the bank and true discount of say $500 for 30 days is only a little over one cent. It would be worth more than 5 cents of the cashier's time to make the longer calculation. Of course the difference would not be so slight in every case, but it should be noted that one and sometimes more than one other person's time besides the cashier's is involved. Then liability to error is much greater in the longer calculation, and this is an important item. Hence it is plain that the method of discounting notes pursued by the banks was not adopted (as the writer once thought before he began making computations like the above) with the object of extorting more money from their customers, but for purely practical reasons.

ALGEBRA.

92. Proposed by **W. F. BRADBURY, A. M.**, Head Master Latin School, Cambridge, Mass.

Find the sum to n terms of $1+3^3+5^3+\ldots.$ [From Charles Smith's *Elementary Algebra*, page 403].

I. Solution by **DR. E. D. ROE, Jr.**, Associate Professor of Mathematics in Oberlin College, P. O., Norwood, Mass.

We have $s_{3,\,n}=1^3+2^3+\ldots.n^3=\left(\dfrac{n\,(n+1)}{2}\right)^2$. (Todhunter's *Algebra*, page 263).

Now $s_{3,2n}=2^3s_{3,\,n}+[1^3+3^3+5^3+\ldots.(2n-1)^3]$.
Hence $1^3+3^3+5^3+\ldots.(2n-1)^3=s_{3,2n}-8s_{3,\,n}=n^2(2n^2-1)$.

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Master and Instructor in Mathematics and Science, Cooper Training School, Carthage, Tex.; COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let $1^3+3^3+5^3+\ldots.(2N-1)^3=A+BN+CN^2+DN^3+EN^4$.
Then $(2N+1)^3=B+(2N+1)C+D(3N^2+3N+1)+E(4N^3+6N^2+4N+1)$.
Equating coefficients, we find $E=2$, $D=0$, $C=-1$, $B=0$.
Therefore $S=A-N^2+2N^4$.
But when $N=1$, $A=0$. $\therefore$ $S=N^2(2N^2-1)$.

III. Solution by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama, University, Alabama; G. B. M. ZEER, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; CHARLES C. CROSS, Libertytown, Md.; J. SCHEFFER, A. M., Hagerstown, Md.; and ELMER SCHUYLER, A. M., High Bridge, N. J.

The general term of the series is $(2n-1)^3$.

$$\therefore\ \Sigma(2n-1)^3=8\Sigma n^3-12\Sigma n^2+6\Sigma n-\Sigma 1$$

$$=8\frac{n^2(n+1)^2}{4}-12\,\frac{n(n+1)(2n+1)}{6}+6\frac{n\,(n+1)}{2}-n=n^2(2n^2-1).$$

IV. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and ELMER SCHUYLER, High Bridge, N. J.

From the well known formula $1^3+2^3+3^3+\ldots.n^3=\dfrac{n^2(n+1)^2}{4}$ we have

$$1^3+2^3+3^3+4^3+\ldots.(2n-2)^3+(2n-1)^3=(2n-1)^2n^2.$$

Denoting the required sum by S, we have,

$S+2^3+4^3+6^3+\ldots.(2n-2)^3=(2n-1)^2n^2$,

or $S+2^3[1+2^3+3^3+\ldots.(n-1)^3]=(2n-1)^2n^2$,

or $S+2^3\,\dfrac{(n-1)^2n^2}{4}=(2n-1)^2n^2$, whence $S=n^2(2n^2-1)$.

V. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; O. S. WESTCOTT, A. M., North Division High School, Chicago, Ill.; ELMER SCHUYLER, High Bridge, N. J.; A.H.BELL, Hillsboro, Ill.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; P. S. BERG, Principal of Schools, Larimore, N. D.; W. L. HARVEY, Portland, Me.; CHARLES E. MEYERS, Canton, O.; ALOIS F. KOVARIK, Decorah Institute, Decorah, Ia.; HAROLD C. FISKE, Union College, Schenectady, N. Y.; and J. SCHEFFER, A. M., Hagerstown, Md.

Solving by the Differential Method, the general formula for the sum of the n terms of a series is

$$Sum=na+\frac{n\,(n-1)}{2}a_1+\frac{n(n-1)(n-2)}{2\times3}a_2+\frac{n(n-1)\,(n-2)\,(n-3)}{2\times3\times4}a_3+\ldots\ldots$$

in which a=first term of series, and a_1, a_2, a_3, a_4, etc., are the respective first terms of the successive orders of differences.

Then 1   27   125   343   729....$(2n-1)^3$=given series.
    26   98   218   386.............=first order of differences.
      72   120   168 ....=second order of differences.
        48   48 ....=third order of differences.
          0..........................=fourth order of differences.

Therefore $a=1$, $a_1=26$, $a_2=72$, $a_3=48$, $a_4=0$.

Substituting these values in the general formula, we obtain

$$Sum=n+13n(n-1)+12n(n-1)(n-2)+2n(n-1)(n-2)(n-3)$$
$$=2n^4-n^2=n^2(2n^2-1)$$

=*the sum of the cubes of the first n odd numbers.*

When $n=4$, $n^2(2n^2-1)=496$.

When $n=5$, $n^2(2n^2-1)=1225=\square$.

COROLLARIES BY M. A. GRUBER.

COROLLARY 1. In a similar manner we find $2^3+4^3+6^3....+(2n)^3=2n^2(n+1)^2$=*the sum of the cubes of the first n even numbers.*

COROLLARY 2. By a similar process we obtain $1+2^3+3^3+4^3+....+n^3=\left[\frac{n(n+1)}{2}\right]^2$; or *the sum of the cubes of the first n natural numbers is equal to the square of the sum of the numbers.* For $1+2+3+4+....+n=\frac{n(n+1)}{2}$.

COROLLARY 3. $1+3^3+5^3+... +(2n-1)^3=n^2(2n^2-1)=\square$, when $n=1$, 5, 29, 169, 985, 5741, etc., or when n=the integral hypothenuse of a right triangle whose legs are consecutive integers. [See AMERICAN MATHEMATICAL MONTHLY, Vol. IV., No. 1, pages 24-27].

COROLLARY 4. $2^3+4^3+6^3+....+(2n)^3$ can never be a square; for $2n^2(n+1)^2$ is *twice a square.*

GEOMETRY.

109. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Two circles, radii in ratio 3:1, centers A and O_1 respectively, are drawn tangent externally to each other and internally to a given circle O, and on the same diameter; O_2 and O_2' are drawn tangent externally to O and internally to A and O_1; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2, A and O_2', respectively, and so on. Prove O_4, O, O_4'; O_5, A, O_5'; O_9, A, O_3' and O_{10}, O, O_2' are collinear. [The letters apply to the centers of the circles].

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Construct the figure as indicated by the problem, and let B be the point of tangency of the circles A and O.

We will solve generally by taking $m:1$ as the ratio of the radii of circles A and O_1.

Put m=radius of circle A. Then 1=radius of circle O_1, $m+1$=radius of circle O, and $\dfrac{m(m+1)}{m(m+1)+(n-1)^2}$=radius of circle O_n, in which n=the number of the smaller circle, beginning with circle O_1.

(a). In $\triangle AO_nO$, taking AO as base,

$$AO_n=m+\frac{m(m+1)}{m(m+1)+(n-1)^2}=\frac{m\,[(m+1)^2+(n-1)^2]}{m(m+1)+(n-1)^2},$$

$$OO_n=m+1-\frac{m(m+1)}{m(m+1)+(n-1)^2}=\frac{(m+1)\,[m^2+(n-1)^2]}{m(m+1)+(n-1)^2},$$

$$AO=1,\ \text{area}=\frac{m(m+1)(n-1)}{m(m+1)+(n-1)^2},\ \text{and altitude}=\frac{2m(m+1)(n-1)}{m(m+1)+(n-1)^2}.$$

(b). We shall now find the value of n when centers O_n, O and O_r' are collinear.

Then $\angle O_r'OO_1=\angle O_nOB$; also OO_r' : sine $\angle O_r'OO_1=OO_n$: sine $\angle O_nOB$.

By substituting values as found in (a), and by using r and n in the proper places, we have

$$\frac{(m+1)\,[m^2+(r-1)^2]}{m(m+1)+(r-1)^3}:\frac{2m(m+1)(r-1)}{m(m+1)+(r-1)^2}$$

$$=\frac{(m+1)\,[m^2+(n-1)^2]}{m(m+1)+(n-1)^2}:\frac{2m(m+1)(n-1)}{m(m+1)+(n-1)^2}.$$

Whence $m^2+(r-1)^2 : r-1=m^2+(n-1)^2 : n-1$;

$m^2+(r-1)^2 : (n-1)^2-(r-1)^2=r-1 : n-r$;

$m^2+(r-1)^2 : n+r-2=r-1 : 1$.

From this we find $n=\dfrac{m^2}{r-1}+1$; also $r=\dfrac{m^2}{n-1}+1$.

When $r=n$, $(n-1)^2=m^2$; whence $n=m+1$.

$r=1$, $n=\infty$.

$r=2$, $n=m^2+1$.

$r=3$, $n=\dfrac{m^2}{2}+1$, m=*even* number.

$r=4$, $n=\dfrac{m^2}{3}+1$, m=multiple of 3.

$r=5$, $n=\dfrac{m^2}{4}+1$, m=*even* number.

etc., etc.

(c). We shall next find the value of n when centers O_n, A and O_r' are collinear.

Following a similar reasoning and substitution as in (b), we obtain

$$n=\frac{(m+1)^2}{r-1}+1\ ;\ \text{also}\ r=\frac{(m+1)^2}{n-1}+1.$$

When $r=n$, $(n-1)^2=(m+1)^2$; whence $n=m+2$.

$r=1$, $n=\infty$.

$r=2$, $n=(m+1)^2+1$.

$r=3$, $n=\dfrac{(m+1)^2}{2}+1$, m=*odd* number.

$r=4$, $n=\dfrac{(m+1)^2}{3}+1$, $m=3p-1$, as 2, 5, 8, 11, etc.

$r=5$, $n=\dfrac{(m+1)^2}{4}+1$, m=*odd* number.

etc., etc.

From (b) and (c) we obtain the following groups of collinear centers, when $m=3$: O_1, O, A, O_∞; O_2', O, O_{10}; O_2', A, O_{17}; O_3', A, O_9; O_4', O, O_4; and O_5', A, O_5.

It will be observed that where a center, in any of the above groups, has a *prime* mark, the mark may be transferred from the first to the last center of the group; as, for group O_2', O, O_{10}, we havs O_2, O, O_{10}'. This is evident from an inspection of the figure.

COROLLARY 1. The figure presents *three varieties of triangles*, which, in general forms, may be represented as AO_nO, $AO_{n+1}O_n$, and $OO_{n+1}O_n$, whose *areas are rational*. Area of triangle AO_nO is shown in (a).

COROLLARY 2. The curve passing through the centers of the smaller circles, (*i. e.* through O_1, O_2, O_3.... $O_\infty[=B]$.... O_3', O_2', O_1), is an ellipse whose foci are A and O. For, $OO_1+AO_1=OO_2+AO_2=OO_n+AO_n=O_1B$.

The general values of the focal radii and ordinate of center O_r are shown in (a).

[NOTE. Mr. Charles C. Cross furnished me with the value $\dfrac{3}{13+n(n-2)}$ =radius of circle O_n, which I found correct according to his notation, and from which I deduced the general value $\dfrac{m(m+1)}{m(m+1)+(n-1)^2}$.]

Excellent solutions with diagrams were received from *G. B. M. ZERR*, *CHARLES C. CROSS*, and *J. SCHEFFER*.

DISCUSSION OF INVERSE FUNCTIONS.

BY COOPER D. SCHMITT, A. M., PROFESSOR OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENN.

1°. NOTATION. If $\sin x = a$, we know that x can be found, and hence in order to express this value of x explicitly we write it $x = \sin^{-1} a$. The first expression means, the sine of the arc, x, is a, and "sine" is the subject, that is, our attention is called to that as being equal to a. The second expression has the arc, x, for its subject, and the second member of the equation is algebraic short hand, telling what arc we mean.

We read it, "x equals the arc whose sine is a," or more briefly, "arc-sine a," *i. e.* x equals arc-sine a.

The expression is sometimes read "x equals the inverse sine of a," or "the anti-sine of a."

Similarly, $\tan 45° = 1$, becomes $45° = \frac{1}{4}\pi = \tan^{-1} 1$,

$\cos 60° = \frac{1}{2}$, becomes $60° = \frac{1}{3}\pi = \cos^{-1}\frac{1}{2}$.

Also, $y = \tan^{-1} x$ can be written $\tan y = x$, and so on. The convention is, that what is inverse on one side is written direct on the other; thus $\sec^{-1} a = x$ becomes $\sec x = a$.

2°. Since $\tan x = \tan x$ we can say $\tan^{-1}(\tan x) = x$ or $\tan(\tan^{-1})x = x$. This becomes self-evident when expressed at length. Thus, the tangent of the arc whose tangent is x is x.

Similarly, $\cos^{-1}(\cos a) = a$, $\sin \sin^{-1} b = b$. The two symbols are not said to cancel each other but to annul or neutralize each other.

3°. There are always two angles less than 360° whose functions equal a certain quantity, but for simplicity, we will always mean the smaller of the two. Thus $\tan^{-1} 1 = 45°$ or $225°$, since the tangent of either of these angles is 1. But 45° is taken unless for some special reason the larger angle is desired.

4°. Any inverse function can be converted into all the other inverse functions.

Thus $\sin^{-1}\frac{3}{5}$ becomes $\tan^{-1}\frac{3}{4}$ by constructing a triangle with hypotenuse 5 and perpendicular 3. From this we see that $\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} = \cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$, etc., etc.

Similarly $\cos^{-1}\frac{2}{3} = \sin^{-1}\frac{\sqrt{5}}{3} = \tan^{-1}\frac{\sqrt{5}}{2}$, etc., etc.

Similarly $\tan^{-1}\frac{5}{12} = \cot^{-1}\frac{12}{5} = \sin^{-1}\frac{5}{13} = \cos^{-1}\frac{12}{13}$.

5°. Any formula of trigonometry expressed in direct notation can be converted into a corresponding formula in the inverse notation.

Thus to convert $\sin 2x = 2\sin x \cos x$ ...(1).

Let $\sin x = a$. Then $\cos x = \sqrt{1-a^2}$ and $x = \sin^{-1} a$.

(1) can be written according to the convention agreed upon in 1°.

$2x = \sin^{-1}(2\sin x \cos x)$. Whence $2\sin^{-1} a = \sin^{-1}(2a\sqrt{1-a^2})$.

Similarly $\cos(x+y) = \cos x \cos y - \sin x \sin y$....(2).

Let $\cos x = a$, then $x = \cos^{-1} a$. Let $\cos y = b$, then $y = \cos^{-1} b$.

And (2) can be written $x+y=\cos^{-1}(\cos x\cos y-\sin x\sin y)$.

Whence $\cos^{-1}a+\cos^{-1}b=\cos^{-1}(ab+\sqrt{1-a^2}\sqrt{1-b^2})\ldots.(3)$,
which is a formula for adding two inverse cosines and getting the result as a single inverse cosine.

$$\text{Again } \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y} \text{ or } x+y=\tan^{-1}\left(\frac{\tan x+\tan y}{1-\tan x\tan y}\right)\ldots(4).$$

Let $\tan x=a$, $\tan y=\tan b$, and (4) becomes

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}\ldots.(5).$$

$$\text{Similarly,} \tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}\ldots.(6).$$

6°. From 4° we see that any inverse function can be expressed as an inverse tangent, and by these last two formulae, (5) and (6), we can add or subtract any two inverse tangents ; hence these two formulae are all that are necessary for combining any number of inverse functions.

$$\text{Thus, } \tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{13}=\tan^{-1}\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}=\tan^{-1}\tfrac{2}{9}\,;$$

$$\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{4}{3}=\tan^{-1}\frac{\frac{3}{4}+\frac{4}{3}}{1-1}=\tan^{-1}\infty=90^\circ=\tfrac{1}{2}\pi\,;$$

$$2\tan^{-1}\tfrac{3}{4}=\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{3}{4}=\tan^{-1}\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{9}{16}}=\tan^{-1}\tfrac{24}{7}.$$

To add $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$ we might use (3), but it is better to convert the $\cos^{-1}$ into $\tan^{-1}$ as in 4°.

$$\text{Thus } \cos^{-1}\tfrac{4}{5}+\cos^{-1}\tfrac{12}{13}=\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{5}{12}=\tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{5}{16}}=\tan^{-1}\tfrac{56}{33}.$$

7°. If fractional coefficients occur they must be gotten rid of.

Thus $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}-\frac{1}{2}\cos^{-1}\frac{3}{5}$ becomes

$$\tfrac{1}{2}(2\tan^{-1}\tfrac{1}{4}+2\tan^{-1}\tfrac{2}{9}-\cos^{-1}\tfrac{3}{5})$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{2}{9}+\tan^{-1}\tfrac{2}{9}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}\left(\tan^{-1}\frac{\frac{1}{4}+\frac{1}{4}}{1-\frac{1}{16}}+\tan^{-1}\frac{\frac{2}{9}+\frac{2}{9}}{1-\frac{4}{81}}-\tan^{-1}\tfrac{4}{3}\right)$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{8}{15}+\tan^{-1}\tfrac{36}{77}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}\left(\tan^{-1}\frac{\frac{8}{15}+\frac{36}{77}}{1-\frac{8}{15}\cdot\frac{36}{77}}-\tan^{-1}\tfrac{4}{3}\right)=\tfrac{1}{2}\left(\tan^{-1}\tfrac{1156}{867}-\tan^{-1}\tfrac{4}{3}\right)$$

$$=\tfrac{1}{2}\left(\tan^{-1}\tfrac{4}{3}-\tan^{-1}\tfrac{4}{3}\right)=0.$$

$4\tan^{-1}\frac{1}{5}$ can be written $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}$ and now the addition formulae can be used at once.

Similarly, for any integral coefficients.

Suppose we are to show that $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

We proceed as follows: $\cos^{-1}\frac{63}{65}=\tan^{-1}\frac{16}{63}$, $\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$, and now we have to show that $\tan^{-1}\frac{16}{63}+2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{3}{4}$, which is done at once as in previous example.

It is thus seen that the formulae $\tan^{-1}a\pm\tan^{-1}b=\tan^{-1}\dfrac{a\pm b}{1\mp ab}$ will apply to most if not all of the examples as given in our text-books and that the work is done entirely in the inverse notation.

CALCULUS.

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84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation to the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

No solution of this problem has yet been received.

It can be easily shown that $ds=(1+\dfrac{\rho}{\rho'})ds'$ where ρ and ρ' are the radii of curvature of the rolling and fixed curves, and ds and ds' arcs of the roulette (the curve generated by the focus of the ellipse) and the pedal curve, the origin of which is the generating point. Finding expressions for ds and ds' and substituting in the above equation and solving for ρ, we have an expression for the curvature of the fixed curve. By substituting for ρ, the general value for the radius of curvature, we derive a very complicated differential equation of the curve. I am not aware that the problem has ever been solved. If any of our contributors will send us a complete solution we shall be pleased to publish it. ED. F.

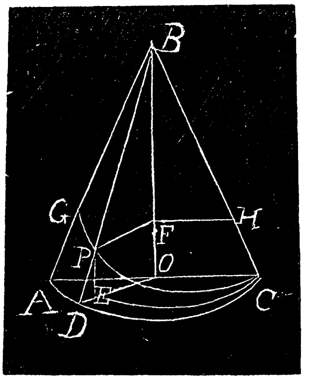
85. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A line of double curvature, beginning at some point in the circumference of the base circle of a right cone, winds itself under the constant inclination β to the base circle around the curved surface of the cone. Find its length and that of its projection upon the base circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be the cone, CPG the line of curvature, σ its length, CE its projection on base, s the length of this projection, BO the z-axis. Also let $DO=CO=a$, $PF=r$, $\angle DPE=$semi-vertical angle of cone$=\gamma$, $\angle PCD=\beta$. Let P, C be

two consecutive points on the curve. Draw PE perpendicular to DO. Then $PC=d\sigma$, $CE=ds$, $PE=dz$, $DE=dr$, $\angle COE=d\theta$, $\angle DPE=\angle DBO=\gamma$.



Then $DE=PD\sin\gamma=PC\sin\gamma\sin\beta$.

$\therefore$ $dr=\sin\gamma\sin\beta d\sigma$.

$\therefore$ $\sigma=\dfrac{r}{\sin\gamma\sin\beta}=\dfrac{a}{\sin\gamma\sin\beta}$, where $a=r$.

$PE=PD\cos\gamma=PC\cos\gamma\sin\beta$.

$\therefore$ $dz=\cos\gamma\sin\beta d\sigma$. $\therefore$ $z=\sigma\cos\gamma\sin\beta$.

$CE=\sqrt{(CP^2-PE^2)}$.

$\therefore$ $ds=\sqrt{(d\sigma^2-dz^2)}=d\sigma\sqrt{(1-\cos^2\gamma\sin^2\beta)}$.

$\therefore$ $s=\sigma\sqrt{(1-\cos^2\gamma\sin^2\beta)}=\dfrac{a\sqrt{(1-\cos^2\gamma\sin^2\beta)}}{\sin\gamma\sin\beta}$

$r.PFH\doteq PH=PC\cos\beta$.

$\therefore$ $rd\theta=\cos\beta d\sigma$.

$\therefore$ $d\theta=\dfrac{\cos\beta d\sigma}{r}=\dfrac{\cot\beta}{\sin\gamma}\cdot\dfrac{d\sigma}{\sigma}$. $\therefore$ $\theta=\dfrac{\cot\beta}{\sin\gamma}\log\sigma$.

$x=r\cos\theta$, $y=r\sin\theta$, $z=\sigma\cos\gamma\sin\beta=r\cot\gamma$ are the equations to the curve. σ and s as given above are the values asked for in the problem.

MECHANICS.

77. Proposed by ELMER SCHUYLER, High Bridge, N. J.

At what elevation must a shell be projected with a velocity of 400 feet that it may range 7500 feet on a plane which descends at an angle of 30°?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

Let $OR=7500$ feet$=R$, $V=400$ feet, $\angle POS=e=$angle of elevation, and $\angle ROS=i=30°$.

$\therefore$ $PR:OP=\sin(e+i):\cos i$.

But $OP=vt$, and $PR=\frac{1}{2}gt^2$.

$\therefore$ $t=2v\sin(e+i)/g\cos i$. $OR:PR=\cos e:\sin(e+i)$.

$\therefore$ $R=gt^2\cos e/2\sin(e+i)=2v^2\sin(e+i)\cos e/g\cos^2 i$.

$$\therefore\ \tan e=\frac{v^2\pm\sqrt{(2v^2gR\sin i-g^2R^2\cos^2 i+v^4)}}{gR\cos i}=\frac{160000\pm143348.62575}{208928.62775}.$$

$\therefore$ $\tan e=1.451925$ or $.079699$.

$\therefore$ $e=55°\ 26'\ 36''$ or $4°\ 33'\ 24''$.

II. Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering and Physics, Agricultural and Mechanical College, College Station, Texas.

If the point of projection be origin and the path be regarded as a parabola its equation will be

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha}\ldots(1),$$

where α=angle of elevation.

At the point where the shell is to strike the plane, $y=-3750$ feet and $x=6495$ feet. Inserting these values in equation (1) there results

$$-3750\times400\cos^2\alpha=6495\times200\times\sin\alpha\cos\alpha-(1299)^2.$$

For $\cos^2\alpha$ write $1+\cos2\alpha$, and for $2\sin\alpha\cos\alpha$ write $\sin2\alpha$, and as $\sin2\alpha=\sqrt{(1-\cos^22\alpha)}$, we get

$$\sqrt{(1-\cos^22\alpha)}=.722-.577\cos2\alpha\ldots(2).$$

Solving (2) we get $\cos2\alpha=314\pm.676$.

$\therefore$ $2\alpha=8^\circ\ 48'$ or $111^\circ\ 12'$ and $\alpha=4^\circ\ 24'$ or $55^\circ\ 36'$, which values satisfy the condition

$$\alpha''-\tfrac{1}{2}(\tfrac{1}{2}\pi-30)=\tfrac{1}{2}(\tfrac{1}{2}\pi-30)-\alpha'.$$

See Tait and Steele's *Dynamics of a Particle*, page 90.

Also solved by *P. H. PHILBRICK.*

78. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A cone and a cylinder having equal heights and equal circular bases are filled with water; if they have equal holes in the bases, respectively, how many times as long will it take the cylinder to empty as the cone?

I. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

Let r=radius of base, h=altitude, k=area of orifice, and x=height of the water at the time t.

I. For the cylinder, the discharge in time dt is, $k\sqrt{(2gx)}dt$; and since in the same time the surface of the water descends a distance dx, the quantity in the vessel is *lessened* πr^2dx.

$\therefore$ $k\sqrt{(2gx)}dt=\pi r^2dx$, and

$$t=-\frac{\pi r^2}{k\sqrt{(2g)}}\int\frac{dx}{\sqrt{x}}=-\frac{2\pi r^2}{k\sqrt{(2g)}}x^{\frac{1}{2}}+c=\frac{2\pi r^2}{k\sqrt{(2g)}}(h^{\frac{1}{2}}-x^{\frac{1}{2}}).$$

II. For the cone, we have $y=(r/h)(h-x)$, and the area of section

$$=\pi y^2=\frac{\pi r^2}{h^2}(h-x)^2.$$

$\therefore k\sqrt{(2gx)}dt=\dfrac{\pi r^2}{h^2}(h-x)^2dx$, and $t=-\dfrac{\pi r^2}{h^2k\sqrt{(2g)}}\displaystyle\int\dfrac{(h-x)^2}{x^{\frac{1}{2}}}dx$

$$=-\frac{\pi r^2}{h^2k\sqrt{(2g)}}\int\frac{(h^2-2hx+x^2)dx}{x^{\frac{1}{2}}}$$

$$=-\frac{\pi r^2}{h^2k\sqrt{(2g)}}.(2h^2x^{\frac{1}{2}}-\frac{4h}{3}x^{\frac{3}{2}}+\tfrac{2}{5}x^{\frac{5}{2}})+c.$$

But $t=0$ for $x=h$. $\therefore c=\dfrac{\pi r^2}{h^2k\sqrt{(2g)}}\tfrac{16}{15}h^{\frac{5}{2}}$.

Hence $t=-\dfrac{\pi r^2}{h^2k\sqrt{(2g)}}(2h^2x^{\frac{1}{2}}-\dfrac{4h}{3}x^{\frac{3}{2}}+\tfrac{2}{5}x^{\frac{5}{2}})+\dfrac{\pi r^2}{k\sqrt{(2g)}}\tfrac{16}{15}h^{\frac{1}{2}}$.

For $x=0$, the time of emptying vessel$=t_2=\dfrac{\pi r^2}{k\sqrt{(2g)}}.\tfrac{16}{15}h^{\frac{1}{2}}$.

This is $\frac{8}{15}$ of the time of emptying the cylinder.

II. **Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.**

Let A=the area of the descending surface, O=the area of the orifice, and x=the depth of the water at the end of any time t.

The quantity of water discharged through the orifice in the infinitely small time dt is $O.dt(2gx)^{\frac{1}{2}}$, the velocity of discharge being $(2gx)^{\frac{1}{2}}$; but in the same time the surface has descended through the distance dx and the quantity discharged is Adx.

$\therefore O.dt(2gx)^{\frac{1}{2}}=Adx$, or $dt=\dfrac{Adx}{O.(2gx)^{\frac{1}{2}}}$, as general formula for any shape of vessel.

Now for the cylinder $A=\pi r^2$ and therefore $dt=-\dfrac{\pi r^2dx}{O(2gx)^{\frac{1}{2}}}$ and

$$t=-\frac{\pi r^2}{O(2g)^{\frac{1}{2}}}\int_h^o\frac{dx}{x^{\frac{1}{2}}}=\frac{2\pi r^2h^{\frac{1}{2}}}{O(2g)^{\frac{1}{2}}}=\frac{\pi r^2(2h)^{\frac{1}{2}}}{O.g^{\frac{1}{2}}};$$

and for the cone $A=\dfrac{\pi r^2(h-x)^2}{h^2}$ and

$\therefore dt=-\dfrac{\pi r^2(h-x)^2dx}{Oh^2(2g)^{\frac{1}{2}}}$ and $t=-\dfrac{\pi r^2}{Oh^2(2g)^{\frac{1}{2}}}\displaystyle\int_h^o\dfrac{(h-x)^2dx}{x^{\frac{1}{2}}}$

$$\frac{\pi r^2}{Oh^2(2g)^{\frac{1}{2}}}.\frac{16h^{\frac{5}{2}}}{15}=\tfrac{8}{15}.\frac{\pi r^2(2h)^{\frac{1}{2}}}{O.g^{\frac{1}{2}}}.$$

Comparing the two results we see that the cone empties in $\frac{8}{15}$ of the time it takes the cylinder, or the cylinder takes $1\frac{7}{8}$ as long as the cone to empty. The minus sign is prefixed because x decreases as t increases.

Also solved by *G. B. M. ZERR*, *ELMER SCHUYLER*, *J. SCHEFFER*, and *J. C. NAGLE*.

NOTE. In reference to problem 63, Dr. Arnold Emch says: "I had the problem solved by my class in graphic statics by a purely graphical method and the following values (approximations) were obtained: $\angle ABE=46°$, $\angle BAD=56°$, tension in $BE=56.8$, tension in $AD=71$. This shows that the solution in the MONTHLY is correct."

DIOPHANTINE ANALYSIS.

71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

By using $(s-1)^2$, the denominator of Euler's fifth number, where $s=4n(n-1)(n+1)[4n(2n-1)(2n+1)]$, I have found five numbers in terms of n: $x=n-1$, $y=n+1$, $z=4n$, $w=4n(2n-1)(2n+1)$, and

$$v=\frac{4n(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1](8n^2-1)}{\{4n(n-1)(n+1)[4n(2n-1)(2n+1)]-1\}^2}.$$

The numerator of v is four times the product of the roots of the six squares $xy+1$, $xz+1$, $yz+1$, $xw+1$, $yw+1$, and $zw+1$.

The denominator of v is the square of $(xyzw-1)$.

Take $n=1$, 2, 3, 4, 5, 6, etc. We then obtain the following sets of five numbers:

$$0,\quad 2,\quad 4,\quad 12,\quad 420;$$

$$1,\quad 3,\quad 8,\quad 120,\quad \frac{777480}{(2879)^2};$$

$$2,\quad 4,\quad 12,\quad 420,\quad \frac{35455980}{(40319)^2};$$

$$3,\quad 5,\quad 16,\quad 1008,\quad \frac{499902480}{(241919)^2};$$

$$4,\quad 6,\quad 20,\quad 1980,\quad \frac{3822388020}{(950399)^2};$$

$$5,\quad 7,\quad 24,\quad 3432,\quad \frac{20000100120}{(2882879)^2};\ \text{etc., etc., etc.}$$

We also find

$$xv+1=\frac{\{(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1]-2n(8n^2-1)\}^2}{(s-1)^2};$$

$$yv+1=\frac{\{(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1]+2n(8n^2-1)\}^2}{(s-1)^2};$$

$$zv+1=\frac{\{16n^4(2n-1)(2n+1)-(8n^2-1)\}^2}{(s-1)^2};\text{ and}$$

$$wv+1=\frac{\{24n^2(2n^2-1)(2n-1)(2n+1)-1\}^2}{(s-1)^2}.$$

74. Proposed by **O. W. ANTHONY, M. Sc.**, Instructor in Mathematics, Boys' High School, New York City.

Solve $x^2+y^2=\square$, $z^2+w^2=\square$, $y^2+w^2=\square$.

I. Solution by **M. A. GRUBER, A. M.**, War Department, Washington, D. C.

Take any two integral equations in which the sum of two squares equals a square, as

$$a^2+b^2=c^2,\text{ and } a_1^2+b_1^2=c_1^2.$$

Multiply the terms of the first equation by the first term and the second term, respectively, of the second equation. Also multiply the terms of the second equation by the first term and the second term, respectively, of the first equation. We then have

$$(aa_1)^2+(a_1b)^2=(a_1c)^2\ldots.(1),$$
$$(ab_1)^2+(bb_1)^2=(b_1c)^2\ldots.(2),$$
$$(aa_1)^2+(ab_1)^2=(ac_1)^2\ldots.(3),$$
$$(a_1b)^2+(bb_1)^2=(bc_1)^2\ldots\ (4).$$

Now put $x=aa_1$, $y=a_1b$, $z=ab_1$, and $w=bb_1$; then equations (1), (2), and (4) are the three required by the problem, there being added, in the solution, $x^2+z^2=\square\ldots.(3)$.

By means of the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, find a few integral numerical equations.

Take $m=2$, $n=1$; then $4^2+3^2=5^2\ldots.(1)$.

Take $m=3$, $n=2$; then $12^2+5^2=13^2\ldots.(2)$.

Take $m=4$, $n=1$; then $8^2+15^2=17^2\ldots.(3)$.

Take $m=5$, $n=2$; then $20^2+21^2=29^2\cdots\cdot(4)$, etc.

From (1) and (2), $x=48$, $y=36$, $z=20$, $w=15$.

From (1) and (3), $x=32$, $y=24$, $z=60$, $w=45$.

From (1) and (4), $x=80$, $y=60$, $z=84$, $w=63$.

From (2) and (3), $x=96$, $y=40$, $z=180$, $w=75$, etc.

IV. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In problems where $x^2+y^2=\square$, $z^2+w^2=\square$, $x^2+z^2=\square$, and $y^2+w^2=\square$, we have the proportion $x:y=z:w$.

Now take two integers the sum of whose squares equals a square, and arrange them in an identical proportion.

Then take two integers of the same kind and arrange them, underneath the first proportion, in an identical proportion of alternation as compared with the first proportion.

Then find the products, term by term, of these two proportions; and the four products will be the required numbers.

Take $3^2+4^2=5^2$, and $5^2+12^2=12^2$.

$$\begin{array}{c} x:y=z:w \\ \hline 3:4=3:4 \\ 3:5=12:12 \\ \hline 15:20=36:48 \end{array}$$

$15^2+20^2=25^2$, $36^2+48^2=60^2$, $15^2+36^2=39^2$, $20^2+48^2=52^2$.

V. Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Manifestly x and y, and z and w, are the bases and perpendiculars of two different right-angled triangles. Hence $x=m^2-n^2$, and $y=2mn$; and $z=p(m^2-n^2)$, and $w=2pmn$. But $y^2+w^2=\square$. Or $4p^2m^2n^2+4m^2n^2=\square$, or $p^2+1=\square=$(say) $(pq-1)^2$. From which $p=\dfrac{2q}{q^2-1}$. Then $z=\dfrac{2q(m^2-n^2)}{q^2-1}$, and $w=\dfrac{2qmn}{q^2-1}$, in which m, n, and q may be any numbers, $q>1$, and $m>n$.

Also solved by *A. H. BELL*, *CHARLES C. CROSS*, *ELMER SCHUYLER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\dfrac{.\dot{2}9\dot{7}}{1.00\dot{3}}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain %?

113. Proposed by B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.

In what time can a note of \$5280, bearing 6% interest, be paid by paying \$600 a year? [Solve by arithmetic].

⁂ Solutions of these problems should be sent to B. F. Finkel not later than June 10.

ALGEBRA.

100. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

Solve, $x^{x+y}=y^{4a}$, $y^{x+y}=x^a$.

101. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Prove that $(1+2+3+\ldots.+n)+\frac{n}{2\,!}(1^2+2^2+3^2+\ \ldots+n^2)+\frac{n\,(n-1)}{3\,!}$ $(1^3+2^3+3^3+\ldots.+n^3)+\frac{n(n-1)(n-2)}{4\,!}(1^4+2^4+3^4+\ldots.+n^4)+\ldots\ldots\ldots$ $+\frac{n(n-1)(n-2)}{4\,!}(1^{n-3}+2^{n-3}+3^{n-3}+\ldots.+n^{n-3})+\frac{n\,(n-1)}{3\,!}(1^{n-2}+2^{n-2}+3^{n-2}$ $+\ldots.+n^{n-2})+\frac{n}{2\,!}(1^{n-1}\ 2^{n-1}\ 3^{n-1}+\ldots.+n^{n-1})+(1^n+2^n+3^n+\ldots.+n^n)$ $=(n+1)^n-1$, and substitute for $n=2$, 3, 4, 5, and 6.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

GEOMETRY.

120. Proposed by P. C. CULLEN, Principal Public Schools, Indianola, Neb.

Draw a circle tangent to a given circle and tangent to a given chord at a given point.

121. Proposed by AUGUSTUS J. REEF, Carbondale, Ind.

Construct a triangle having given its three medians. [From Wentworth's *Plane and Solid Geometry*].

122. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, Manchester High School, Manchester, N. H.

If perpendiculars are dropped from the vertices of a regular polygon upon any diameter of the circumscribed circle, the sum of the perpendiculars which fall on one side of this diameter is equal to the sum of those which fall on the opposite side. [From Chauvenet's *Treatise on Elementary Geometry*.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

CALCULUS.

90. Proposed by ELMER SCHUYLER, High Bridge, N. J.

Prove that the evolute of the logarithmic spiral is an equal logarithmic spiral. [From Byerly's *Integral Calculus*].

91. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Find the area of a loop of the curve $r^2\cos\theta=a^2\sin3\theta$. [From Hall's *Differential and Integral Calculus*].

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

MECHANICS.

89. Proposed by **GUY B. COLLIER**, Schenectady, N. Y.

Assuming that the Northern Pacific R. R. tracks between Fargo and Bismark (North Dakota) to lie on the 47th parallel of latitude; also that the Limited Express weighs 300 tons, and that a speed of 60 miles per hour is maintained between the two places; find the difference between the vertical pressures on the rails of the Express east and the Express west.

90. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

Adopting the hypothesis that the planets were originally all one mass revolving about a fixed center and were formed by an explosion of this mass at some point in its path; prove that, if the law of nature were that force varies directly as the distance, the planets would all have collided again simultaneously, and find an expression for the time between the explosion and collision.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than June 10.

DIOPHANTINE ANALYSIS.

80. Proposed by **M. A. GRUBER, A. M.**, War Department, Washington, D. C.

Find three square numbers whose reciprocals form an arithmetical progression.

⁂ Solutions of this problem should be sent to J. M. Colaw not later than June 10.

AVERAGE AND PROBABILITY.

73. Proposed by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School, Chester, Pa.

On an average 1 vessel out of every n is wrecked. Find the chance that out of m vessels expected p at least will arrive safely.

74. Proposed by **F. ANDEREGG, A. M.**, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

From a point in the circumference of a circular field a projectile is thrown at random with a given velocity which is such that the diameter of the field is equal to the greatest range of the projectile. Find the chance of its falling into the field. [From Byerly's *Integral Calculus*, page 209].

⁂ Solutions of these problems should be sent to B. F. Finkel not later than June 10.

MISCELLANEOUS.

77. Proposed by **T. E. COLE**, Columbus, Ohio.

It is said that a base-ball pitcher throws curves. Give a scientific explanation of how it is done.

78. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

The center of a regular polygon of n sides moves along a diameter of a given circle, the plane of the polygon being perpendicular to the diameter, and its magnitude varying

in such a manner that one of its diagonals always coincides with a chord of the circle; find the surface and the volume generated, and thence deduce the formulae for the surface and the volume of a sphere.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than June 10.

EDITORIALS.

Editor Finkel was elected a member of the London Mathematical Society on April 13th.

THE UNIVERSITY OF CHICAGO, SUMMER, 1899. The following Mathematical Courses will be offered: By Professor *Maschke*, Theory of Functions of a Complex Variable, Abstract Groups; Professor *Hathaway*, Quaternions, Plane Analytics; Assistant Professor *Young*, Conferences on the Pedagogy of Mathematics, Determinants; Assistant Professor *Skinner*, College Algebra; Dr. *Slaught*, Differential Equations, Differential Calculus; Dr. *Boyd*, Twisted Curves, Solid Geometry.

BOOKS.

An Introduction to the Differential and Integral Calculus and Differential Equations. By F. Granville Taylor, M. A., B. Sc., Mathematical Lecturer at University College, Nottingham. 8vo. Cloth, 592 pages. Price, 8*s*. London and New York: Longmans, Green & Co.

This work comprises three sections, in the first of which is given a very thorough treatment of the Differential Calculus and its applications; the second is devoted to treatment of the Integral Calculus; and the third deals with the elementary methods of solving Ordinary Differential Equations of the first and second orders. In the Differential and Integral Calculus, the author has given a few practical applications as early as possible, in order that the beginner may have some notion of the uses to which the Calculus may be put. Curve Tracing receives a good deal of attention; Hyperbolic Functions and their differentiation has received due consideration. Throughout the work, numerous examples are given, these being well selected and graded in a way to stimulate and inspire the student. The subject of the Calculus as presented in this work is clear and simple, and is a worthy rival of the many valuable works on this subject. B. F. F.

A Text-book of General Physics for the use of Colleges and Scientific Schools. By Charles S. Hastings, Ph. D., and Frederic E. Beach, Ph. D., of Yale University. 8vo. Half Leather Back, v+768 pages. Price, $2.95. Boston: Ginn & Co.

In this book, the fact is emphasized that a knowledge of Elementary Mechanics is the logical basis of the whole science of Physics. With this in view, we find here a more complete treatment of Mechanics than is ordinarily the case, especially in the physical notions which attach to the simplest cases of the action of force. Numerous problems are appended to the various chapters, the solutions of which will go far towards impressing the principles upon the mind of the student. The method of presenting the subject of Physics as here given is very good. B. F. F.

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ON SYMMETRIC FUNCTIONS.

By E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from April Number.]

5. FORMULAS FOR THE FOUR KINDS OF REDUCTION.

We will place here together the four formulas of reduction. They are :

$$\begin{pmatrix} 0^{\lambda_0} \; 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \; \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0-\lambda} \; 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^{\kappa_1} \; \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix}, \quad m-\lambda=\kappa_1,$$

$$\begin{pmatrix} 0^{\lambda_0} \; 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \; \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{n\kappa_n} \begin{pmatrix} 0^{\lambda_0} \; 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n-\kappa_n} \\ 0^{m-\kappa_n} \; (\varkappa_1-\varkappa_n)(\varkappa_2-\varkappa_n)\ldots(\varkappa_{n-1}-\varkappa_n) \end{pmatrix},$$

$$\begin{pmatrix} 0^{\lambda_0} \; 1^{\lambda_1} \ldots r(r+1)^0 \ldots n^0 \\ 0^m \; \varkappa_1 \; \varkappa_2 \ldots\ldots\ldots \varkappa_r 0^{n-r} \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0} \; 1^{\lambda_1} \ldots\ldots r^{\lambda_r} \\ 0^m \; \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_r \end{pmatrix},$$

$$\begin{pmatrix} r^{\lambda_r} \; (r+1)^{\lambda_{r+1}} \ldots n^{\lambda_n} \\ 0^m \; \varkappa_1 \; \varkappa_2 \ldots\ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{\kappa_1 r} \begin{pmatrix} 0^{\lambda_r} \; 1^{\lambda_{r+1}} \ldots (n-r)^{\lambda_n} \\ 0^{\kappa_1} \; \varkappa_{r+1} \; \varkappa_{r+2} \ldots\ldots \varkappa_n \end{pmatrix}, \quad \varkappa_1=\varkappa_2=\ldots=\varkappa_r=m.$$

E. DERIVATION.

We will next give the four formulas of derivation, corresponding in order to those of reduction.

1. The First Formula.

The first formula of derivation is

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m & \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n\end{pmatrix}=\begin{pmatrix}0^{\lambda_0+\lambda} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^{m+\lambda} & \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n\end{pmatrix},$$

and is seen to be true in that we can multiply both the term and $b_0{}^m\Sigma$ by $b_0{}^\lambda$.

2. The Second Formula.

By B, 1, (2),

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m & \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n\end{pmatrix}=(-1)^{mn}\begin{pmatrix}0^{\lambda_n} & 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m & (m-\varkappa_n)(m-\varkappa_{n-1}) \ldots\ldots (m-\varkappa_1)\end{pmatrix}.$$

By the first formula of derivation 1, and by B, 1, (2),

$$\begin{pmatrix}0^{\lambda_n} & 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m & (m-\varkappa_n)(m-\varkappa_{n-1}) \ldots\ldots (m-\varkappa_1)\end{pmatrix}=\begin{pmatrix}0^{\lambda_n+\kappa} & 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^{m+\kappa} & (m-\varkappa_n)(m-\varkappa_{n-1}) \ldots\ldots (m-\varkappa_1)\end{pmatrix}$$

$$=(-1)^{(m+\kappa)n}\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots\ldots\ldots n^{\lambda_n+\kappa} \\ 0^{m+\kappa} & (\varkappa_1+\varkappa) \ldots\ldots (\varkappa_n+\varkappa)\end{pmatrix}.$$

Therefore by substitution,

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0_m & \varkappa_1 \; \varkappa_2 \ldots\ldots \kappa_n\end{pmatrix}=(-1)^{n\kappa}\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n+\kappa} \\ 0^{m+\kappa} & (\varkappa_1+\kappa)(\varkappa_2+\varkappa) \ldots\ldots (\varkappa_n+\varkappa)\end{pmatrix},$$

and this is the second formula of derivation.

3. The Third Formula.

The third formula of derivation comes from the third formula of reduction in that we turn the latter about, and write n^1 instead of r, $n^1+\varkappa$ instead of n, and consequently $\varkappa$ instead of $n-r$. We then have, dropping accents,

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m & \varkappa_1 \; \varkappa_2 \ldots\ldots \varkappa_n\end{pmatrix}=\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n}(n+1)^0 \ldots\ldots (n+\varkappa)^0 \\ 0^m & \varkappa_1 \; \varkappa_2 \ldots\ldots\ldots\ldots\ldots\ldots \varkappa_n \; 0^\kappa\end{pmatrix}.$$

4. The Fourth Formula.

We can obtain the fourth formula of derivation from the fourth of reduction by turning the latter about. Remembering that $\varkappa_1=\varkappa_2=\ldots\ldots=\varkappa_r=m$, the fourth formula of reduction may be written

$$\begin{pmatrix}r^{\lambda_r} & (r+1)^{\lambda_{r+1}} \ldots\ldots n^{\lambda_n} \\ 0^m & m^r \; \varkappa_{r+1} \ldots\ldots\ldots \varkappa_n\end{pmatrix}=(-1)^{mr}\begin{pmatrix}0^{\lambda_r} & 1^{\lambda_{r+1}} \ldots\ldots (n-r)^{\lambda_n} \\ 0^m & \varkappa_{r+1} \ldots\ldots\ldots\ldots \varkappa_n\end{pmatrix};$$

we may now put,

$$\lambda_r = \lambda_0^1 \qquad \varkappa_{r+1} = \varkappa_1^1$$

$$\lambda_{r+1} = \lambda_1^1 \qquad \varkappa_{r+2} = \varkappa_2^1$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$\lambda_n = \lambda^1{}_{n-r} \qquad \varkappa_n = \varkappa^1{}_{n-r}\text{; we get,}$$

$$\begin{pmatrix} 0^{\lambda_0{}^1}\, 1^{\lambda_1{}^1} \ldots\ldots (n-r)^{\lambda^1{}_{n-r}} \\ 0^m\, \varkappa_1{}^1\, \varkappa_2{}^1 \ldots\ldots\ldots \varkappa^1{}_{n-r} \end{pmatrix} = (-1)^{mr} \begin{pmatrix} r^{\lambda_0{}^1} (r+1)^{\lambda_1{}^1} \ldots\ldots n^{\lambda^1{}_{n-r}} \\ 0^m\, m^r \varkappa_1{}^1\, \varkappa_2{}^1 \ldots\ldots \varkappa^1{}_{n-r} \end{pmatrix}.$$

Finally putting $n-r=n^1$, $n=n^1+r$, and dropping all accents,

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m\, \varkappa_1\, \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{mr} \begin{pmatrix} r^{\lambda_0} (r+1)^{\lambda_1} \ldots\ldots (n+r)^{\lambda_n} \\ 0^m\, m^r\, \varkappa_1\, \varkappa_1 \ldots\ldots\ldots \varkappa_n \end{pmatrix},$$

and this is the fourth formula of derivation.

5. The Four Formulas of Derivation.

We collect the four formulas of derivation. They are:

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m\, \varkappa_1\, \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0+\lambda}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^{m+\lambda}\, \varkappa_1\, \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix}$$

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m\, \varkappa_1\, \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-)^{n\kappa} \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n+\kappa} \\ 0^{m+\kappa} (\varkappa_1+\varkappa)(\varkappa_2+\varkappa) \ldots\ldots (\varkappa_n+\varkappa) \end{pmatrix}$$

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m\, \varkappa_1\, \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots n^{\lambda_n} (n+1)^0 \ldots (n+\varkappa)^0 \\ 0^m\, \varkappa_1\, \varkappa_2 \ldots\ldots\ldots\ldots\ldots \varkappa_n\, 0^\kappa \end{pmatrix}$$

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m\, \varkappa_1\, \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{mr} \begin{pmatrix} r^{\lambda_0} (r+1)^{\lambda_1} \ldots (n+r)^{\lambda_n} \\ 0^m m^r\, \varkappa_1\, \varkappa_2 \ldots\ldots\ldots \varkappa_n \end{pmatrix}.$$

These four kinds of derivation here correspond to the third, first, second, and fourth kinds of derivation, respectively, in the resultant theory.

F. THE CONJUGATE PRODUCT AND FUNCTION.

1. Definitions.

With respect to the product $(a_{r_1})^{p_1}(a_{r_2})^{p_2} \ldots (a_{r_\mu})^{p_\mu}$, the function

$$a_0{}^n \Sigma (a r_1)^{p_1+p_2+\ldots\ldots p_\mu} [\alpha(r_2-r_1)]^{p_2+p_3+\ldots\ldots p_\mu} [\alpha(r_3-r_2)]^{p_3+p_4+\ldots\ldots p_\mu} \ldots\ldots$$

$$[\alpha(r_\mu - r_{\mu-1})]^{p_\mu}$$

$$= a_0{}^n \Sigma (\alpha_1 \alpha_2 \ldots \alpha_r)^{p_1} (\alpha_1 \alpha_2 \ldots \alpha_{r_2})^{p_2} (\alpha_1 \alpha_2 \ldots \alpha_{r_3})^{p_3} \ldots (\alpha_1 \alpha_2 \ldots \alpha_{r_\mu})^{p_\mu},$$

is called the conjugate symmetric function, and with respect to the function, the product is called the conjugate product.

The number n is the order of the function and $n=p_1+p_2+\dots p_\mu$. If the function is given as

$$a_0^{\kappa_1}\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\dots\alpha_n^{\kappa_n}=a_0^{\kappa_1}\Sigma\alpha_1^{\kappa_1-\kappa_2}(\alpha_1\alpha_2)^{\kappa_2-\kappa_3}(\alpha_1\alpha_2\alpha_3)^{\kappa_3-\kappa_4}\dots(\alpha_1\alpha_2\dots\alpha_n)^{\kappa_n},$$

it is clear that the conjugate product is,

$$a_1^{\kappa_1-\kappa_2}\,a_2^{\kappa_2-\kappa_3}\,a_3^{\kappa_3-\kappa_4}\dots a_n^{\kappa_n}.$$

2. The Coefficient of the Conjugate Product in a Symmetric Function. Completely Reducible Forms.

It will next be proved that the coefficient of the conjugate product in a symmetric function is completely reducible, and is equal to $(-1)^w$, where w is the weight of the function. According to 1, this coefficient may be written :

$$\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\dots(n-1)^{\kappa_{n-1}-\kappa_n}\,n^{\kappa_n}\\ 0^{\kappa_1}\,\varkappa_1\,\varkappa_2\,\dots\dots\dots\dots\dots\dots\varkappa_n\end{pmatrix}.$$

By D, 2 (second reduction), we may reduce this, as follows :

$$\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\dots(n-1)^{\kappa_{n-1}-\kappa_n}\,n^{\kappa_n}\\ 0^{\kappa_1}\,\varkappa_1\,\varkappa_2\,\dots\dots\dots\dots\dots\dots\varkappa_n\end{pmatrix}$$

$$=(-1)^{n\kappa_n}\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\dots\dots\dots\dots(n-1)^{\kappa_{n-1}-\kappa_n}\\ 0^{\kappa_1-\kappa_n}(\varkappa_1-\varkappa_n)(\varkappa_2-\varkappa_n)\dots(\varkappa_{n-1}-\varkappa_n)\end{pmatrix}$$

$$=(-1)^{n\kappa_n+(n-1)(\kappa_{n-1}-\kappa_n)}\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\dots\dots\dots\dots\dots(n-2)^{\kappa_{n-2}-\kappa_{n-1}}\\ 0^{\kappa_1-\kappa_{n-1}}(\varkappa_1-\varkappa_{n-1})(\varkappa_2-\varkappa_{n-1})\dots(\varkappa_{n-2}-\varkappa_{n-1})\end{pmatrix}.$$

We have made here two reductions, one after the other. Proceeding in this way, we can now see that after n reductions of the same kind, we obtain,

$$\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\dots\dots n^{\kappa_n}\\ 0^{\kappa_1}\,\varkappa_1\,\varkappa_2\dots\dots\dots\varkappa_n\end{pmatrix}$$

$$=(-1)^{n\varkappa_n+(n-1)(\varkappa_{n-1}-\varkappa_n)+(n-2)(\varkappa_{n-2}-\varkappa_{n-1})+\dots(\varkappa_1-\varkappa_2)}\binom{1^0}{0^0}$$

$$=(-1)^{\kappa_1+\kappa_2+\dots\kappa_n}\binom{1}{1}=(-1)^w.$$

This is the completely reducible form in the theory of symmetric functions. It may also be written as

$$\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots n^{\lambda_n}\\ 0^m(\lambda_1+\lambda_2+\dots\lambda_n)(\lambda_2+\lambda_3+\dots\lambda_n)(\lambda_3+\lambda_4+\dots\lambda_n)\dots(\lambda_{n-1}+\lambda_n)\lambda_n\end{pmatrix}.$$

3. Correspondence of the Conjugate Product with the Completely Reducible Term of $R_{m,n}$.

It will yet be shown more in detail that the conjugate product in a symmetric function corresponds to the completely reducible term in the resultant theory. According to the method of A, 5, the resultant of f and ϕ may be expressed as

$$\Sigma(-1)^{mn+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^m\Sigma(\beta p_1)^{m-r_1}(\beta p_2)^{m-r_2}\ldots(\beta p_\mu)^{m-r_\mu},$$

where g_1 is the weight of $(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}$. [$r_1, r_2 \ldots r_\mu$ are supposed to be arranged in the order of ascending magnitude]. A group of terms will be represented by

$$(-1)^{mn+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^m\Sigma(\beta p_1)^{m-r_1}\ldots(\beta p_\mu)^{m-r_\mu}.$$

Among these terms will be the one containing the conjugate product of the symmetric function involved. It will be

$$(-1)^{mn+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^m\frac{(-1)^{mn-g_1}}{b_0{}^{m-r_1}}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_1}$$

$$\ldots(b_{p_1+p_2+\ldots p_\mu})^{m-r_\mu}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{r_1}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_1}$$

$$\ldots(b_{p_1+p_2+\ldots p_\mu})^{m-r_\mu}.$$

But this by A, 1, (2) is the completely reducible term of $R_{m,n}$.

G. RECURRENCE FORMULA FOR THE COEFFICIENT

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\ldots\ldots\ldots\ldots n^{\lambda_n} \\ 0^m m^{\mu_0}(m-1)^{\mu_1}(m-2)^{\mu_2}\ldots 0^{\mu_m}\end{pmatrix}.$$

1. Statement of Requirement.

Since all the coefficients of terms in symmetric functions can be reduced to such as are normal forms, or else are completely reducible and have the coefficient $(-1)^w$, we require farther only a means of calculating the normal forms. We will obtain a recurrence formula for the normal form

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots\ldots n^{\lambda_n} \\ 0^m m^{\mu_0}(m-1)^{\mu_1}\ldots 0^{\mu_m}\end{pmatrix},\ \text{where } m\geqq n.$$

2. Derivation of the Formula.

From the equation $\phi=0$, we have

$$b_0\beta_1{}^m+b_1\beta_1{}^{m-1}+b_2\beta_1{}^{m-2}+\ldots+b_r\beta_1{}^{m-r}+\ldots+b_n\beta_1{}^{m-n}=0.$$

We multiply this equation by

$$(b_0)^{m-1}(\beta_2\beta_3\ldots\beta_{\mu_0})^m(\beta_{\mu_0+1}\ldots\beta_{\mu_0+\mu_1})^{m-1}\ldots(\beta_{\mu_{m-2}+1}\ldots\beta_{\mu_0+\mu_1+\ldots\mu_{m-1}})^1$$

$(\beta_{\mu_{m-1}+1}\ldots\beta_{\mu_0+\ldots\mu_m})^0$ and get $(b_0)^m(\beta\mu_0)^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\ldots$

$$(\beta\mu_m)^0+(b_0)^{m-1}b_1[\beta(\mu_0-1)]^m[\beta(\mu_1+1)]^{m-1}\ldots(\beta\mu_m)^0+\ldots$$

$$+(b_0)^{m-1}b_r[\beta(\mu_0-1)]^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\ldots[\beta(\mu_r+1)]^{m-r}\ldots(\beta\mu_m)^0+\ldots=0.$$

We will next take the sum of all such equations, using all possible substitutions in the summation. We obtain a series of symmetric functions, of which the first contains $\dfrac{m!}{\mu_0!\,\mu_1!\,\mu_2!\,..\,\mu_m!}$ terms, and the $(r+1)$st $\dfrac{m!}{(\mu_0-1)!\,\mu_1!\,\mu_2!\,..(\mu_r+1)!\,..\mu_m!}$ terms. These numbers can be written as $\dfrac{1}{\mu_0}\lambda$, and $\dfrac{1}{\mu_r+1}\lambda$, where $\lambda=\dfrac{m!}{(\mu_0-1)!\,\mu_1!\,\mu_2!\,..\mu_r!\,..\mu_m!}$.

But by this mode of summation, the first function will be repeated $\dfrac{n!}{\frac{1}{\mu_0}\lambda}=\mu_0\dfrac{n!}{\lambda}$ times, and the $(r+1)$st function $(\mu_r+1)\dfrac{n!}{\lambda}$ times.

We get therefore, by our summation, dividing by the common factor $n!/\lambda$,

$$\mu_0 b_0^m\Sigma(\beta\mu_0)^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\ldots(\beta\mu_r)^{m-r}\ldots(\beta\mu_m)^0+$$

$$(b_0)^{m-1}\sum_{r=1}^{r=n}b_r(1+\mu_r)\Sigma[\beta(\mu_0-1)]^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\,..[\beta(\mu_r+1)]^{m-r}\,..(\beta\mu_m)^0\equiv 0.$$

If in this equation we pick out the coefficient of $b_0^{\lambda_0}b_1^{\lambda_1}b_2^{\lambda_2}\ldots b_n^{\lambda_n}$ we get,

$$\mu_0\begin{pmatrix}0^{\lambda_0}\;1^{\lambda_1}\;2^{\lambda_2}\ldots\ldots n^{\lambda_n}\\ 0^m m^{\mu_0}(m-1)^{\mu_1}..0^{\mu_m}\end{pmatrix}=-\sum_{r=1}^{r=n}(1+\mu_r)\begin{pmatrix}0^{\lambda_0}\;1^{\lambda_1}\;2^{\lambda_2}\ldots\ldots r^{\lambda_r-1}\ldots\ldots n^{\lambda_n}\\ 0^{m-1}m^{\mu_0-1}(m-1)^{\mu_1}..(m-r)^{\mu_r+1}..0^{\mu_m}\end{pmatrix}$$

$$=-\sum_{r=1}^{r=n}(1+\mu_r)\begin{pmatrix}0^{\lambda_0+1}\;1^{\lambda_1}\;2^{\lambda_2}\ldots\ldots r^{\lambda_r-1}\ldots\ldots n^{\lambda_n}\\ 0^m\;m^{\mu_0-1}(m-1)^{\mu_1}..(m-r)^{\mu_r+1}..0^{\mu_m}\end{pmatrix}.$$

3. Correspondence of the Formula with the Recurrence Formula in the Resultant Theory.

The preceding formula of 2 corresponds to the recurrence formula of A, 4, in the resultant theory. The identity of the two can be shown by eliminating the coefficients of the resultant from the latter formula by means of the relation [A, 5, (2)]

$(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n) \mid 0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\ldots.n^{\lambda_n}$

$$=(-1)^{\kappa_1+\kappa_2+\ \ldots\kappa_n}\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1}\ldots\ldots n^{\lambda_n}\\ 0^m & \varkappa_1\ \ \varkappa_2\ldots.\varkappa_n\end{pmatrix}$$

after suitably changing the notation, or we may first write the formula of 2 as

$$c_0\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1}\ldots\ldots n^{\lambda_n}\\ 0^m & \varkappa_1\ \ \varkappa_2\ldots.\varkappa_n\end{pmatrix}=-\sum_{r=1}^{r=n}(1+c_r)\begin{pmatrix}0^{\lambda_0+1} & 1^{\lambda_1}\ldots\ldots\ldots n^{\lambda_n}\\ 0^m & (\varkappa_1-r)\ \varkappa_2\ldots.\varkappa_n\end{pmatrix},$$

where at least $\varkappa_1=m$, $\varkappa_n=0$, and c_r is the number of β's which have the exponent $m-r$.

4. The Case where $n>m$.

In case $n>m$ we may use one of the relations

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1}\ldots\ldots n^{\lambda_n}\\ 0^m & \varkappa_1\ \ \varkappa_2\ldots.\varkappa_n\end{pmatrix}=\begin{pmatrix}\varkappa_1\,\varkappa_2\ldots\ldots\ldots\varkappa_n\\ 0^n\ 0^{\lambda_0}\ 1^{\lambda_1}\ldots.n^{\lambda_n}\end{pmatrix}$$

$$=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n)\\ 0^n0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ \ldots\ldots\ldots\ldots n^0\end{pmatrix}$$

of B, 1, (2), and then compute as before by the reduction formula of 2.

[To be Concluded.]

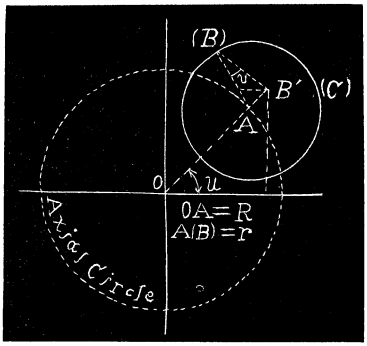
NOTE ON THE LOXODROMIC LINES OF THE TORUS.*

By Dr. ARNOLD EMCH, Professor of Graphic Mathematics, Kansas State Agricultural College, Manhattan, Kas.

1. Designating by u and v the angles, which, in Fig. 1, determine the position of a point B on the surface of a torus, the square of a linear element on the surface has the form,

$$ds^2=(R+r\sin v)^2(du_1^2+dv_1^2) \quad \ldots\ldots(1),$$

$$\text{where } u_1=u \text{ and } v_1=\int_0^v \frac{dv}{\frac{R}{r}+\sin v} \quad \ldots\ldots(2).$$



(C)—Revolved Meridian.

Fig. 1.

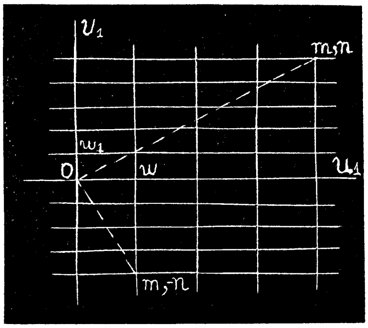


Fig. 2.

R and r are respectively the radii of the axial circle and of a meridian of the torus. As it is well known, by means of the expressions (2), the points of the torus are conformally transformed into the points of a rectangle. Considering exclusively the case where $R>r$ the integral v_1 has the value

$$v_1=\frac{r}{\sqrt{R^2-r^2}}\text{arc sin}\left(\frac{1+\frac{R}{r}\sin v}{\frac{R}{r}+\sin v}\right)\ldots\ldots(3).$$

The values of u and v can each vary from 0 to 2π. In this interval u_1 varies from 0 to 2π and according to the expression (3), v_1 from

$$\frac{r}{\sqrt{R^2-r^2}}\text{arc sin}\,\frac{r}{R}\text{ to }\frac{r}{\sqrt{R^2-r^2}}\left[\text{arc sin}\,\frac{R}{r}+2\pi\right].$$

*A preliminary statement concerning orthographic loxodromics on a torus was made by the author in a paper, "Ueber orthogonale Systeme und einige technische Anwendungen," which appeared in the program of the Polytechnic of Biel, 1898.

The sides of the primitive rectangle are therefore

$$w=2\pi \text{ and } w_1=\frac{r2\pi}{\sqrt{R^2-r^2}} \ldots\ldots(4).$$

To the lines u_1=const., v_1=const., respectively, parallel to the sides of the rectangle (w, w_1), correspond on the torus the meridians and parallels, and vice versa to the meridians and parallels u=const. and v=const., correspond in the rectangle lines, respectively, parallel to the sides w and w_1.

2. Evidently these lines form isothermal systems. If we now put $u_1+iv_1=z$, and if α and β designate two complex quantities $\alpha=p-iq$, $\beta=r(1+i)$, the monogenic function

$$\varphi+i\psi=\alpha z+\beta$$

represents also an isothermal system in the (u_1, v_1)-plane. There is

$$\varphi=au_1+bv_1+c\ldots\ldots(5),$$
$$\psi=-bu_1+av_1+c\ldots\ldots(6),$$

which represent two perpendicular pencils of parallel rays. Interpreted by means of formulae (2) they represent two systems of orthographic loxodromic lines on the torus.

3. Among the loxodromic lines of the torus we shall consider those that close after a certain number of revolutions around the axis and the axial circle of the torus. For this purpose it will be well to refer to the integral

$$Z=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}\ldots\ldots(7),$$

by which the positive part of the z-plane is transformed into the rectangle of the z-plane whose sides correspond to the periods of the elliptic function

$$z=\lambda(Z) \ \ldots\ .(8),$$

as defined by the elliptic integral (7).*

In order that the sides of the rectangle assume the values w and w_1, the modulus k of integral (7) must be chosen in such a manner that according to a well-known formula

$$k=4\sqrt{q}\left[\frac{(1+q^2)(1+q^4)(1+q^6)\ldots\ldots}{(1+q)(1+q^3)(1+q^5)\ldots\ldots}\right]^4,$$

where $q=e^{-(2\pi w_1)/w}$.

*See F. Klein, Ueber Riemann's Theorie der Algebraische Funktionen und ihrer Integrale, pages 50—55.

Supposing that $z=\lambda(Z)$ has been determined in this way, it is possible to transform the surface of the torus conformally upon the positive part of the z-plane and to express the periodic character of this transformation. The general period of $z=\lambda(Z)$ has the form

$$W=mw+niw,$$

or also

$$W=2\pi\left(m+ni\frac{r}{\sqrt{R^2-r^2}}\right)\ldots\ldots(9).$$

By means of this expression for the period it is easy to show an interesting relation which exists between two orthographic loxodromic lines of a torus. From the origin O of the system of parallelograms of periods draw two lines, one to the point $\left(2m\pi,\ 2ni\pi\frac{r}{\sqrt{R^2-r^2}}\right)$; other to point $\left(2m_1\pi,\ -2n_1 i\pi\frac{r}{\sqrt{R^2-r^2}}\right)$ of the system, (Fig. 2). The trigonometric tangents of these lines with the positive part of the u_1-axis are

$$u=\frac{n}{m}\cdot\frac{r}{\sqrt{R^2-r^2}},\quad v=-\frac{n_1}{m_1}\cdot\frac{r}{\sqrt{R^2-r^2}},$$

and the tangent of the angle included by the two lines is

$$\frac{\dfrac{n}{m}\cdot\dfrac{r}{\sqrt{R^2-r^2}}+\dfrac{n_1}{m_1}\cdot\dfrac{r}{\sqrt{R^2-r^2}}}{1-\dfrac{n}{m}\cdot\dfrac{n_1}{m_1}\cdot\dfrac{r}{\sqrt{R^2-r^2}}}.$$

The condition for a right angle is

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{r}{\sqrt{R^2-r^2}}\ldots\ldots(10).$$

In this case the corresponding loxodromic lines on the torus are orthographic and since in the plane (u_1, v_1) the lines end in points of periods, it is clear that the two orthographic loxodromic lines close. If the numbers m and n are given, and provided R/r is a rational fraction, then it is always possible to find two integers m_1 and n_1 so that relation (10) is satisfied. Evidently m gives the number of revolutions around the axis (perpendicular to the parallels) and n the number of revolutions around the axial circle of the torus. Thus we are led to the theorem :

If the ratio R/r of a torus is a rational fraction and if a loxodromic line of the torus closes after m revolutions around the axis and n revolutions around the

axial circle of the torus, then every orthographic loxodromic line closes after m_1 *and* n_1 *revolutions around the same axes, so that*

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{r}{\sqrt{R^2-r^2}}.$$

4. Among the great number of special cases, which arise from this relation, we shall simply mention the case where $m=n=1$. The following theorem is easily found:

The loxodromic lines of a torus, which turn once around the axis and once around the axial circle of the torus, are the circles formed by the intersection of the double-tangent planes with the torus.

The results found about loxodromic lines of a torus may immediately be generalized for those cyclides which arise from the torus by circular transformations.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

108. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A man who feels his death approach bequeaths to his young wife one-third of his fortune, and the remaining two-thirds to his son, if such should be born; but one-half of it to the widow and the other half to his daughter, if such should be born. After his death, twins are born, a son and a daughter. How should the fortune be divided amongst the three?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; P. S. BERG, B. S., Principal of Schools, Larimore, N. D.; SYLVESTER ROBINS, North Branch, N. J.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; ELMER SCHUYLER, High Bridge, N. J.; and ALOIS F. KOVARIK, Professor of Mathematics and Physics, Decorah Institute, Decorah, Ia.

By the conditions of the problem, the wife is to have as much as the daughter and half as much as the son.

$\therefore$ son : wife : daughter $=2:1:1$.

$\therefore$ son should receive $\frac{1}{2}$ the fortune, wife should receive $\frac{1}{4}$ the fortune, and daughter should receive $\frac{1}{4}$ the fortune.

II. Solution by D. G. DORRANCE, Camden, N. Y.

Consider the fortune (a) divided into two fortunes each of one-half the value of the whole, viz: $\frac{1}{2}a$ and $\frac{1}{2}a$.

Then widow's share of first $\frac{1}{2}a$ would be $\frac{1}{3}$ of $\frac{1}{2}a=\frac{1}{6}a$, and the son's share would be $\frac{2}{3}$ of $\frac{1}{2}a=2a/6$ or $\frac{1}{3}a$.

Widow's share of second $\frac{1}{2}a$ would be $\frac{1}{2}$ of $\frac{1}{2}a=\frac{1}{4}a$, and the daughter's share would be $\frac{1}{2}$ of $\frac{1}{2}a=\frac{1}{4}a$.

$\therefore$ Widow should receive $\frac{1}{6}a+\frac{1}{4}a=5a/12$. Son should receive $4a/12$. Daughter should receive $3a/12$.

[See Vol. I., page 127, for solution of a similar problem. Also see Cajori's *History of Mathematics*, page 79. ED. F.]

109. Proposed by **B. F. FINKEL, A. M., M. S.,** Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Why do fences and telegraph poles appear to move rapidly in an opposite direction to one traveling in a railway car? [From Moore's *Grammar School Arithmetic*, page 150.]

I. Answer by the **PROPOSER.**

When we are at rest and observing a moving object, a line from our eye to the object projects the object in different points on the landscape, or, if the object is in the sky, as, for example, a cloud, the line from our eye to the cloud projects the cloud in different points of the sky. By noticing the different projected positions of the object, we become conscious of its motion. Now if we are moving and are looking at some stationary object, the line from our eye to the object again projects the object in different points of the landscape, the appearance being the same as in the case where the object moved and we were stationary. But in this case the apparent motion of the object is opposite to our real motion.

When we are unconscious of our own motion, as, for example, when riding in a car over a smooth road, we attribute the appearance of an object projected in different positions on the landscape to the motion of the object, and since our motion is direct, the apparent motion of the object is retrograde. This motion is more striking as the object upon which our attention is fixed is the nearer, for the reason that the angle of parallax is greater.

II. Answer by **G. B. M. ZERR, A. M., Ph. D.,** Professor of Science and Mathematics, Chester High School, Chester, Pa.

When we are standing on the street watching a moving object, the line from the eye to the object is constantly changing its direction causing us to turn our head in order to follow the object with our eye. When we are sitting in a railway train, the train is moving with us in it and seems to us as if it were still and the objects passing the same as when we were standing on the street. In either case we must turn the head, hence we seem to see the objects passing us instead of our passing the objects.

III. Answer by **ALOIS F. KOVARIK,** Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

On the train we are stationary with respect to the cars, but in motion with respect to the earth. Noticing the things next to us not to be in motion, while

the posts, etc., outside the car seem passing by on account that we judge mostly by what is near our eyes. It is this that being stationary with respect to the earth, the ancients imagined the sun and stars to move around the earth instead of the opposite.

GEOMETRY.

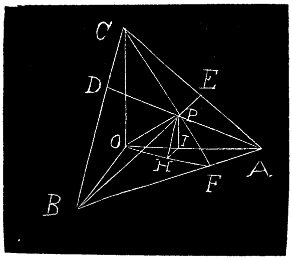
110. Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

If the three face angles of the vertical triedral angle of a tetraedron are right angles, and the lengths of the lateral edges are represented by a, b, and c, and of the altitude by p, then $1/p^2=1/a^2+1/b^2+1/c^2$. [Chauvenet's *Geometry*.]

I. Solution by GEORGE R. DEAN, Professor of Mathematics, School of Mines and Metallurgy, University of Missouri, Rolla, Mo.; P. H. PHILBRICK, C. E., Lake Charles, La.; and CHAS. C. CROSS, Libertytown, Md.

Let OC, OB, OA be the edges mutually at right angles; OP the perpendicular. Join AP and produce to meet CB at D; join BP and produce to meet AC at E; join CP and produce to meet AB at F.

Then $$\frac{\triangle APC}{\triangle ABC}+\frac{\triangle BPC}{\triangle ABC}+\frac{\triangle APB}{\triangle ABC}=1.$$



But $$\frac{\triangle PAC}{\triangle ABC}=\frac{PE}{BE},\quad \frac{\triangle BPC}{\triangle ABC}=\frac{DP}{AD},$$

$$\frac{\triangle BPA}{\triangle ABC}=\frac{PF}{CF}.$$

Therefore $$\frac{PE}{BE}+\frac{DP}{AD}+\frac{PF}{CF}=1.$$

$$BE\times BP=OB^2=b^2;\quad BE=\frac{b^2}{BP};\quad \frac{PE}{BE}=\frac{PE\times BP}{b^2}=\frac{OP^2}{b^2}.$$

$$AD\times AP=OA^2=a^2;\quad AD=\frac{a^2}{AP};\quad \frac{DB}{AD}=\frac{AP\times DP}{a^2}=\frac{OP^2}{a^2}.$$

$$CF\times CP=OC^2=c^2;\quad CF=\frac{c^2}{CP};\quad \frac{PF}{CF}=\frac{PF\times PC}{c^2}=\frac{OP^2}{c^2}.$$

Hence $$\frac{OP^2}{a^2}+\frac{OP^2}{b^2}+\frac{OP^2}{c^2}=1,\text{ or }\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.\qquad \text{Q. E. D.}$$

II. Solution by P. S. BERG, B.Sc., Superintendent of Schools, Larimore, N. D.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ELMER SCHUYLER, High Bridge, N. J.; W. H. WILSON, Professor of Mathematics, Geneva College. Beaver Falls, Pa.; and G. B. M. ZERR, A. M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $OP=p$, $OC=a$, $OA=b$, $OB=c$, $OI=l$, $IH=m$, $PH=n$.

The triangles (right angled) OPA and OPI are similar, having $\angle POA$ in common.

$\therefore p/b=l/p$, or $p^2/b^2=l^2/p^2$, or $p^4/b^2=l^2$.

Similarly, $p^4/a^2=n^2$, $p^4/c^2=m^2$.

$\therefore p^4/a^2+p^4/b^2+p^4/c^2=n^2+l^2+m^2=p^2$.

$\therefore 1/a^2+1/b^2+1/c^2=1/p^2$.

Also solved by *J. SCHEFFER*, and *E. D. SCALES*.

111. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Given that the area of a triangle is equal to half the product of two sides and the sine of the included angle, prove that $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

I. Solutfon by W. F. BRADBURY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass., and B. F. YANNEY, A. M., Professsor of Mathematics, Mount Union College, Alliance, O.

Given area $\triangle=\frac{1}{2}ac\sin B$. . . . .(1).

$\sin B=\sin[180^\circ-(A+C)]=\sin(A+C)$.

$\therefore$ Area $\triangle=\frac{1}{2}ac[\sin(A+C)]$. . . . . .(2).

Draw BD perpendicular to AC.

Area $\triangle=\frac{1}{2}BD(AD+DC)$.

But $BD=c\sin A=a\sin C$, and $AD=c\cos A$, and $DC=a\cos C$.

$\therefore$ Area $\triangle=\frac{1}{2}\begin{matrix}c\sin A\\ \text{or}\\ a\sin C\end{matrix}(c\cos A+a\cos C)=\frac{1}{2}(ac\sin C\cos A+ac\sin A\cos C)$. . .(3).

Putting (2)=(3), $\frac{1}{2}ac[\sin(A+C)]=\frac{1}{2}ac(\sin A\cos C+\cos A\sin C)$ or $\sin(A+C)=\sin A\cos C+\cos A\sin C$.

II. Solution by J. OWEN MAHONEY, B. E., Professor of Mathematics and Science, Carthage High School, Carthage, Tex.; J. W. YOUNG, Columbus, O.; J. SCHEFFER, A. M., Hagerstown, Md ; E. L. SHERWOOD, A. M., Professor of Mathematics, Whiteworth College, Miss.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; JOHN MACHNIE, A. M., Professor of Latin, University of North Dakotå ; H. F. STRATTON, Student in Heidelberg University, Tiffin, O.; J. C. NAGLE, C. E., Professor of Civil Engineering and Physics in the Agricultural and Mechanical College of Texas, College Station, Tex.; CHARLES C. CROSS, Libertytown, Md.; and ELMER SCHUYLER, High Bridge, N. J.

Proof. Consider the triangle ACB.

The area of $ACB=\frac{1}{2}ac\sin B=\frac{1}{2}ac\sin(A+C)=\frac{1}{2}hb=\frac{1}{2}h(AD+DC)$, or

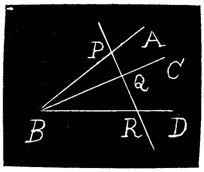
$$\sin(A+C)=\frac{h}{a}\cdot\frac{AD}{c}+\frac{h}{c}\cdot\frac{DC}{a}$$

$$=\sin C\cos A+\sin A\cos C.$$

$$\therefore \sin(x+y)=\sin x\cos y+\cos x\sin y.$$

III. Solution by the PROPOSER.

Let ABC and CBD be the two angles x and y. Draw any line perpendicular to BC meeting AB, BC, BD at P, Q, R.



Then $\triangle BPR=\triangle BPQ+\triangle BQR$.

$2\triangle BPR=BP\times BR\times \sin PBR=BP.BR.\sin(x+y)$.

$2\triangle BPR=BP\times BQ\times \sin PBQ=BP.BQ.\sin x$.

$2\triangle BQR=BQ.RB.\sin y$.

Hence $BP.BR\sin(x+y)=BP.BQ\sin x+BQ.BR\sin y$.

Dividing by $BP.BR$, $\sin(x+y)=\frac{BQ}{BR}\sin x+\frac{BQ}{BP}\sin y=\cos y\sin x+\cos x\sin y$.

CALCULUS.

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84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation of the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Let θ be the angle between the major axis of the ellipse and the radius vector from the focus to the point of contact; r the length of this radius, a, b, semi-axes, e eccentricity. Then

$$r=\frac{a(1-e^2)}{1-e\cos\theta}.$$

Let the axis of x be taken parallel to the given line. Then since the point of contact is the instantaneous center, the radius vector will always be perpendicular to the axis of x, and hence $r+y=a(1-e)$, y being the ordinate of a point on the required curve. We have also

$$r\frac{d\theta}{dr}=-\frac{dx}{dy}.$$

Eliminating r and θ, we find

$$y-\sqrt{b^2\left(\frac{dy}{dx}\right)^2+a^2}=ae.$$

Whence $\dfrac{ae-y}{b}=\sin\left(\dfrac{x}{b}+\sin^{-1}\dfrac{ae}{b}\right)$, or $y'=b\sin\left(\dfrac{x'}{b}\right)$.

86. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove that the curve whose normal equals its radius of curvature drawn in an opposite direction, is the catenary $y=c\cosh(x/c)$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

We should have

$$\frac{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=y\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}},\text{ whence}\frac{2\dfrac{dy}{dx}\dfrac{dy^2}{dx^2}}{1+\dfrac{dy^2}{dx^2}}=\frac{2}{y}\frac{dy}{dx}.$$

Multiplying by dx and integrating and correcting,

$$1+\frac{dy^2}{dx^2}=\frac{y^2}{c^2}.$$

This gives $dx=\dfrac{cdy}{\sqrt{y^2-c^2}}$; whence integrating and correcting again,

$$x=c\log\left(\frac{y+\sqrt{(y^2-c^2)}}{c}\right),\text{ or } y=c\cosh(x/c).$$

II. Solution by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

Our differential equation, formed by substituting the values of normal and ρ, the radius of curvature is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}\text{......(1).}$$

$$\therefore y\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]\text{......(2).}$$

Now let $p=\dfrac{dy}{dx}$, then $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}=\dfrac{dp}{dy}\cdot\dfrac{dy}{dx}=\dfrac{dp}{dy}.p.$

Substituting, $y\,p\,\dfrac{dp}{dy}=(1+p^2)$......(3).

$\therefore\dfrac{pdp}{1+p^2}=\dfrac{dy}{y}$. Integrating,

$\log y=\frac{1}{2}\log(p^2+1)+\log c_1$, or $\log(y/c_1)=\log\sqrt{(p^2+1)}$......(4).

$\therefore y/c_1=\sqrt{(p^2+1)}$......(5).

Squaring and clearing, $c_1p^2=y^2-c_1^2$......(6).

$$p^2=\frac{y^2-c_1^2}{c_1^2}, \text{ or } p=\frac{\sqrt{(y^2-c_1^2)}}{c_1^2}\ldots\ldots(7).$$

Then, substituting for p, and separating,

$$\frac{dy}{\sqrt{y^2-c_1^2}}=\frac{dx}{c_1^2}\ldots\ldots(8). \text{ Integrating, } \log[y+\sqrt{(y^2-c_1^2)}]=\frac{x}{c_1^2}+\log c_2\ \ldots.(9).$$

$$\therefore\ \frac{y+\sqrt{(y^2-c_1^2)}}{c_2}=e^{x/c_1}\ldots\ldots(10).$$

Transposing and squaring, $y^2-c_1^2=c_2^2e^{2x/c_1}-c_2e^{x/c_1}y+y^2\ldots\ldots(11)$.

$$\therefore\ y=\frac{c_2^2e^{2x/c_1}+c_1^2}{c_2 2e^{x/c_1}}\ldots.\ .(12), \text{ or } y=\frac{c_2^2e^{x/c_1}+c_1^2e^{-(x/c_1)}}{2c_2}\ldots\ldots(13).$$

Let $c_2=c_1$, thus moving the origin to the right. Then $y=\dfrac{c_1[e^{x/c_1}+e^{-(x/c_1)}]}{2}$ .. .(14), or $y=c_1\cosh(x/c_1)$, which is the equation of the catenary.

Also solved by *J. W. YOUNG.*

87. **Proposed by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.**

Integrate $(px-y)(py+x)=h^2p$, where $p=dy/dx$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

We have $(px-y)(py+x)=h^2p\ldots\ldots(1)$, or

$$p^2xy-py^2+px^2-xy=h^2p\ldots\ldots(2).$$

Multiplying by y and arranging,

$$y^2=\left(y\frac{dy}{dx}\right)^2-\left(\frac{y^3}{x^2}-xy+\frac{h^2y}{x}\right)\frac{dy}{dx}\ldots\ldots(3).$$

Putting $y^2=y'$, $x^2=x'$,

$$y'=x'\left(\frac{dy'}{dx'}\right)^2-y'\frac{dy'}{dx'}+x'\frac{dy'}{dx'}-\frac{dy'}{dx'}h^2\ldots\ldots(4),$$

or $y'=p'x'-\dfrac{h^2p'}{p'+1}\ldots\ldots(5)$, Clairaut's Form, giving $y^2-cx^2=-\dfrac{ch^2}{c+1}\ \ldots..(6)$.

II. Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Divide by p, differentiate and reduce to

$$xy\frac{dp}{dx}+(px-y)p=0.$$

This may be written

$$\frac{\frac{d}{dx}(px-y)}{px-y}=-\frac{1}{y}\frac{dy}{dx},$$

the integral of which is $pxy-y^2=c$.

Eliminating p between this and the given equation,

$$y^2+\frac{c}{c-h^2}=-c.$$

Putting $\frac{c}{c-h^2}=-c'$, $y^2-c'x^2=\frac{c'h^2}{1+c'}$.

This solution may also be obtained by solving $pxy-y^2=c$, obtaining $y^2+c'x^2=c$.

This gives $p=-\frac{c'x}{y}$, which substituted in the given equation makes $c=\frac{-c'h^2}{1+c'}$.

Also solved by the *PROPOSER*.

MECHANICS.

79. **Proposed by WALTER H. DRANE, Graduate Student, Harvard University. Cambridge, Mass.**

The four wheels of a street car are rigidly fixed to their axles so that axles and wheels turn together. Is it more advantageous to apply the brakes to the front or to the rear wheels, supposing the brakes to block the wheels in each case?

Solution by the PROPOSER.

The question may be answered by treating the problem as a statical one, thus: Suppose the car placed upon an inclined plane and let us inquire in which case may the plane be raised to the greater angle before slipping begins. Let $2a$ be the distance between the centers of the wheels, each of radius c, b the distance of the center of gravity, G of the car above (or below) this line of centers, w the weight of each of the trucks, w_1 the weight of car.

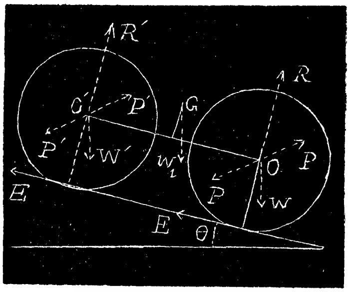
Take the case first when brakes are applied to rear wheels, there being in this case, of course, no friction between the front wheels and plane. Consider the figure as consisting of two rigid bodies, viz, the front trucks, and the car with the rear trucks. The forces acting upon the front trucks are their weight w, the reaction R of the plane, and a force, P, at O obliquely downward, which is the resultant of w and the backward pull of the car and rear wheels. Upon the car at this point O there will also be an equal and opposite force to P, the resultant of R and w.

The forces on car and rear wheels are, this force P at O, w_1 at G, w at O', R_1 the reaction of the plane, and F the force of friction.

From forces on front trucks we have,

$$\left.\begin{array}{l}(1).\ w\sin\theta=P\cos\phi\\(2).\ w\cos\theta=R-P\sin\phi\end{array}\right\}\ \phi \text{ being the angle between } P \text{ and the line of centers } OO'.$$

From forces on car and rear wheels, letting μ be coefficient of friction, taking moments about O' and resolving horizontally and vertically,



(3). $2aP\sin\phi=w_1(a+b\tan\theta)\cos\theta-\mu R_1c.$

(4). $\mu R_1=(w+w_1)\sin\theta+P\cos\phi.$

(5). $R_1+P\sin\phi=(w+w_1)\cos\theta.$

From these five equations we get,

$$\tan\theta=\frac{a\mu(2w+w_1)}{aw_1b+(2a-c\mu)(2w+w_1)}\ldots\ldots(A).$$

Next take the case when brakes are applied to front wheels, considering in this case, the rear trucks as one rigid body, and the car with front trucks as the other. The two oblique forces P now act at O' one upon the car and one upon rear wheels (lettered P' to avoid confusion).

From forces on rear wheels we get,

$$\left.\begin{array}{l}(a).\ w\sin\theta=P'\cos\phi'\\(b).\ w\cos\theta=R_1-P'\sin\phi'\end{array}\right\}\ \phi' \text{ being the angle between } P' \text{ and } OO'.$$

From forces on car and front wheels, taking moments about O, resolving horizontally and vertically.

(c). $2aP'\sin\phi'=w_1(a-b\tan\theta)\cos\theta-\mu Rc.$

(d). $\mu R=(w+w_1)\sin\theta+P'\cos\phi'.$

(e). $R+P'\sin\phi'=(w+w_1)\cos\theta.$

From these five equations we get

$$\tan\theta=\frac{a\mu(2w+w_1)}{(2a-\mu c)(2w+w_1)-\mu w_1b}\ldots\ldots(B).$$

Comparing (B) with (A) we see that in the latter case $\tan\theta$, and hence θ, is the greater, and hence we infer it is more advantageous to apply the brakes to the front wheels.

80. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A circular board is placed on a smooth horizontal plane and a boy runs with uniform speed around on the board close to the edge. Find the motion of the center of the board.

I. Solution by **GEORGE R. DEAN**, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The system is not acted upon by any external force. Hence the center of gravity is stationary. Let I be the moment of inertia of the board about its center, m the mass of the board, m' that of the boy, r the distance of center of system from center of board, ω the angular velocity of board. Then $I\omega^2+mr^2\omega^2=m'v^2$.

II. Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $AC=a$ be the radius of board, A the starting point of the boy, m his mass, P the mass of the board, G the center of gravity of the system at starting.

Since there are no horizontal forces *external* to the system, by D'Alembert's principle, the center of gravity of the system must move downwards ; but this is impossible.

$\therefore$ The equation of motion of the center of gravity is, $m\dfrac{d^2\bar{x}}{dt^2}=0$.

$\therefore \dfrac{d\bar{x}}{dt}=0$, since both boy and board start from rest.

$\therefore \bar{x}=$a constant. $\therefore$ The center of gravity of the system remains unaltered throughout the motion.

Hence since the distance between the center of the board and boy remain constant, the distances of the center of gravity from the boy and the center of the board respectively will remain constant.

$\therefore C$ describes a circle with $GC=\dfrac{ma}{m+P}$ as radius.

$\therefore A$ describes a circle with $AG=\dfrac{Pa}{m+P}$ as radius.

$\therefore$ both center of board and boy describe circles. If $m=P$ these circles will be equal and have AC for diameter.

MISCELLANEOUS.

69. Proposed by **WILLIAM SYMMONDS**, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O., Sebastopol, Cal.

Find the locus of a point equi-distant from the circumferences of two fixed circles.

I. Solution by **H. C. WHITAKER**, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Denote the two given circumferences by K_1 and K_2; take their centers at $(-\frac{1}{2}a, 0)$ and $(\frac{1}{2}a, 0)$; denote their radii by r_1 and r_2, r_1 not being less than r_2. Let (x, y) denote a point in the desired locus.

Then if (x, y) is inside K_1 and inside K_2,

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2-\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=0\ldots\ldots(1).$$

Or if (x, y) is outside K_1 and outside K_2,

$$\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-r_1-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=0,$$

which is the same as (1).

Similarly, if (x, y) is inside K_1 and outside K_2, or if (x, y) is outside K_1 and inside K_2,

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=0\ldots\ldots(2).$$

The desired locus is represented by the equation obtained by taking the product of (1) and (2). I am not aware of any general criterion as to the nature and limiting points of curves (or pieces of curve) whose equations are irrational. If either variable in the equation is an explicit function of the other, the nature of the curve, as well as the limiting points, can usually be detected by inspection; but considerable difficulty may be experienced in finding the nature and limiting points of the curve represented by an irrational equation in which x and y are implicit functions of each other.

Although aware that the method is exceedingly unsatisfactory, I will first include the curves or pieces of curve represented by the equations congeneric to (1) and (2) and afterward exclude such pieces thus incorrectly dragged in by the process of rationalization.

The equations, whose product make (1) rational, are:

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2-\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=0\ldots\ldots(1).$$

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2+\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=A\ldots\ldots(3).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2-\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=B\ldots\ldots(4).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2+\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=C\ldots\ldots(5).$$

The equations, whose product make (2) rational, are

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=0\ldots\ldots(2).$$

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}+\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}+r_2\}=D\ldots\ldots(6).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=E\ldots\ldots(7).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}+\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}+r_2\}=F\ldots\ldots(8).$$

The product of (1), (3), (4), and (5) will represent all lines or parts of lines the points of which satisfy *any one of them with its right hand member put equal to zero*: for if the right hand member of any of them, say (3) was zero, and the right hand member of (1) was, say G, precisely *the same product would be*

obtained ; a similar statement may be made with reference to the product of (2), (6), (7), and (8), although in the case of (8) no points would be obtained by making the right hand member equal to zero, since the sum of two positive distances cannot equal zero.

Denote the difference of the radii (r_1-r_2) by d, and the sum (r_1+r_2) by s. The product of the first four equations is

$$4(a^2-d^2)x^2-4d^2y^2=d^2(a^2-d^2)\ldots\ldots(9).$$

And the product of the second four equations is

$$4(a^2-s^2)x^2-4s^2y^2=s^2(a^2-s^2)\ldots\ldots(10).$$

It will be remembered that a is the distance between the centers of the two given circles, d the difference, and s the sum of their radii. In eliminating from (9) the points not satisfying (1), and from (10) the points not satisfying (2), I will consider a to vary.

I. $a<d<s$ or K_2 is wholly inside K_1 ; in this case both (9) and (10) are equations of ellipses ; (9) is the locus of centers of circles tangent internally to K_1 and enclosing K_2 ; these points satisfy (3) put equal to zero. Equation (10) is the locus of centers of circles tangent internally to K_1 and externally to K_2 ; all such points satisfy (2) and hence (10) is the equation of the desired locus.

II. If $a=d$ and $a<s$, K_2 is inside K_1 and internally tangent to it ; in this case, equation (9) reduces to $y=0$ or the axis of X. From $x=+\infty$ to $x=+\frac{1}{2}a$, the points satisfy (1) ; from $x=+\frac{1}{2}a$ to $x=-\frac{1}{2}a$, the points satisfy (3) put equal to zero ; from $x=-\frac{1}{2}a$ to $x=-\infty$, the points satisfy (5) put equal to zero. Equation (10) as in Case I. The locus consists then of (9), from $x=+\infty$ to $x=+\frac{1}{2}d$, and of (10).

III. If $a>d$ and $a<s$, K_1 and K_2 intersect; in this case equation (9) represents a hyperbola, all the points on the right hand branch satisfying (1), and the points on the left hand branch satisfying (5) put equal to zero. Equation (10) as in Cases I and II. The locus consists of (9), $x=+\infty$ to $x=+\frac{1}{2}d$, and of (10).

IV. If $a<d$ and $a=s$, K_1 and K_2 are tangent externally. Equation (9) as in Case III. Equation (10) reduces to $y=0$, or the axis of X. From $x=+\infty$ to $x=+\frac{1}{2}a$, the points satisfy (6) ; from $x=+\frac{1}{2}a$ to $-\frac{1}{2}a$, the points satisfy (2) ; and from $x=-\frac{1}{2}a$ to $-\infty$, the points satisfy (7). The locus consists of (9) from $x=+\infty$ to $x=+\frac{1}{2}d$, and of (10) from $x=+\frac{1}{2}a$ to $x=-\frac{1}{2}a$.

V. If $a>d$ and $a>s$, K_1 and K_2 are exterior to each other. Equation (9) as in Cases III and IV. Equation (10) also represents a hyperbola, the points on the right hand branch satisfying (6) put equal to zero, and the points on the left hand branch satisfying (7) put equal to zero. The locus consists of (9) from $x=+\infty$ to $x=+\frac{1}{2}d$.

In many cases the limiting points may be found by equating the radical to zero if it is of the second degree as in this case. But this is not true as a gener-

al proposition. For the right hand branch of the hyperbola (9) in III, IV, and V crosses the circle at $x=+\frac{1}{2}a$ and is inside from $x=+\frac{1}{2}a$ to $x=+\frac{1}{2}d$. The left hand branch begins at $x=-\frac{1}{2}d$ and leaves the circle at $x=-\frac{1}{2}a$. Between the limits suggested by equating the radicals to zero ($x=+\frac{1}{2}a$ and $x=-\frac{1}{2}a$) there are two pieces of curve, one of which contains points filling the given conditions, the other containing points violating them. That the tentative methods adopted by me are wholly unsatisfactory I freely admit, and yet I feel sure that graphical methods will be of great aid in establishing a theory of irrational equations.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

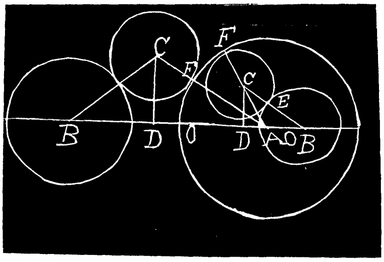
Let $AF=R$, $BE=r$, $CF=CE=t$.

Then when B is inside of circle A we have $BC+AC=r+t+R-t=R+r$.

$\therefore$ The locus of C, the point equidistant from both circumferences, is an ellipse, foci A, B.

When B is without A we have $AC-BC=R+t-r-t=R-r$.

$\therefore$ The locus of C is a hyperbola, foci A, B. Also as follows:



Let the mid-point O, between A and B be origin, $AB=2a$, $OD=x$, $CD=y$.

$\therefore$ $BC^2=(r+t)^2=(a\pm x)^2+y^2 \dots (1)$.

$AC^2=(R\mp t)^2=(x\mp a)^2+y^2 \dots (2)$.

Eliminating t between (1) and (2) we get

$$r\pm R\mp\sqrt{[(x\mp a)^2+y^2]}=\sqrt{[(x\pm a)^2+y^2]}, \text{ or } \frac{4x^2}{(R\pm r)^2}+\frac{4y^2}{(R\pm r)^2-4a^2}=1.$$

This equation shows that when one circle is within the other $(R+r)>2a$ and the locus is an ellipse with semi-axes $\frac{1}{2}(R+r)$ and $\frac{1}{2}\sqrt{[(R+r)^2-4a^2]}$.

When circles are external the locus is an hyperbola with semi-axes $\frac{1}{2}(R-r)$ and $\frac{1}{2}\sqrt{[4a^2-(R-r)^2]}$.

If the circles intersect, for internal contact to one circle and external to the other we have the same ellipse; for both internal and both external we have the same hyperbola as above.

Also solved by *GUY B. COLLIER*, *CHARLES C. CROSS*, *WALTER H. DRANE*, *ALOIS F. KOVARIK*, *W. W. LANDIS*, *J. SHEFFER*, and *COOPER D. SCHMITT*.

DIOPHANTINE ANALYSIS.

73. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Find integral values for x and y in $\begin{pmatrix} 2x^2-y^2=\square \\ 2y^2-x^2=\square \end{pmatrix}$.

Discussion by W. T. KING, Ottawa, Canada.

We may suppose x and y to have no common factor ; for if they have one it will also be a common factor of a and b, and its square may be divided out of both equations, leaving equations of the same form. For the same reason no two of x, y, a, or of x, y, b can have a common factor, nor can a and b, without its being common to all. We therefore suppose that x, y, a and b are all prime to one another. Also we may assume without loss of generality that $x>y$. Then it is evident that $a>x$, $b<y$.

From the first equation we have

$$2(a^2-x^2)=a^2-y^2.$$

Suppose $\dfrac{2(a-x)}{a-y}$ when reduced to its lowest terms to be equal to p/q, so that p and q are prime to one another.

Then $2q(a-x)=p(a-y)$ and $p(a+x)=q(a+y)$.

Whence $(p^2+2q^2)a+(p^2-2q^2)x=2pqa$ or $x/a=\dfrac{p^2+2q^2-2pq}{2q^2-p^2}$.

The numerator being evidently positive, the denominator $(2q^2-p^2)$ must be also.

Since x/a is in its lowest terms, the fraction $\dfrac{p^2+2q^2-2pq}{2q^2-p^2}$, when reduced to its lowest terms, must have its numerator equal to x, and its denominator to a. But it is evident that p and q being prime to one another, the numerator and denominator can have no common factor, unless it be 2, in the event of p being even. Therefore if p be odd,

$$x=p^2+2q^2-2pq\ldots\ldots(1).$$

$$y=2q^2-p^2\ldots\ldots(2).$$

Whence by the equation $q(a+y)=p(a+x)$ we get

$$y=4pq-2q^2-p^2\ \ldots\ .(3).$$

If p be even,

$$x=\tfrac{1}{2}(p^2+2q^2-2pq)=2(\tfrac{1}{2}p)^2+q^2-2(\tfrac{1}{2}p)q,$$

$$a=\tfrac{1}{2}(2q^2-p^2)=q^2-2(\tfrac{1}{2}p)^2,$$

$$y=\tfrac{1}{2}(4pq-2q^2-p^2)=4(\tfrac{1}{2}p)q-q^2-2(\tfrac{1}{2}p)^2.$$

These equations are of the same form as the preceding, except in the case of a. For x and y we may take them in all cases,

$$\left.\begin{aligned}x&=p^2+2q^2-2pq\\ y&=4pq-2q^2-p^2\end{aligned}\right\}\ldots\ldots(4).$$

Now from the equation $2y^2-x^2=b^2$, we have

$$2(y^2-b^2)=x^2-b^2.$$

Whence as before we have,

$$2s(y-b)=r(x-b), \quad r(y+b)=s(x+b),$$

r and s being prime to one another. Whence as before,

$$\left.\begin{array}{l} \left.\begin{array}{l} x=4rs-2s^2-r^2 \\ y=2s^2+r^2-2rs \end{array}\right\} \text{ or quantities of the same form} \\ \qquad\qquad\qquad\qquad\qquad\quad \text{when } r \text{ is even,} \\ b=2s^2-r^2, \text{ or } =s^2-2(\tfrac{1}{2}r)^2, \text{ when } r \text{ is even,} \end{array}\right\} \ldots\ldots(5).$$

Equating the two values of x and y we have $2pq=2rs$.

$\therefore\ x-pq=x-rs$ or $p^2+2q^2-3pq=3rs-2s^2-r^2$ or $(2q-p)(q-p)=(2s-r)(r-s)$.

Now from the equation $p(a+x)=q(a+y)$ we see that $q>p$, for $x>y$.
Then $q-p$ and $2q-p$ are both positive.
Similarly from the equation $r(y+b)=s(x+b)$, we see that $r>s$.
And since $(2s-r)(r-s)$ is positive, $2s-r$ is also positive.
$\therefore\ r>s<2s$. Now we may write

$$\left.\begin{array}{l} n(q-p)=m(r-s) \\ m(2q-p)=n(2s-r) \end{array}\right\} \ldots\ldots(6).$$

where m and n are positive integers, prime to one another.

$$\therefore\ (2m^2+n^2)q-(m^2+n^2)p=mns,$$
$$2(m^2+n^2)q-(m^2+2n^2)p=mnr.$$

$\therefore\ [(2m^2+n^2)q-(m^2+n^2)p][2(m^2+n^2)q-(m^2+2n^2)p]=m^2n^2rs=m^2n^2pq$, or
$2(m^2+n^2)(2m^2+n^2)q^2-2(2m^2+n^2)(m^2+2n^2)pq+(m^2+n^2)(2n^2+m^2p^2=0$.

Since the value of q/p derived from this equation must be rational,

$$(2m^2+n^2)^2(m^2+2n^2)-2(m^2+n^2)(2m^2+n^2)(m^2+2n^2)$$

must be a square, *i. e.* $(2m^2+n^2)(m^2+2n^2)m^2n^2$ must be a square, and $(2m^2+n^2)(m^2+2n^2)$ must be a square.

Therefore either $2m^2+n^2$ and m^2+2n^2 are both squares, or when divided by a common factor, the quotients are squares.

In the first case let $2m^2+n^2=h^2$, $m^2+2n^2=k^2$, then $m^2+n^2=\frac{1}{3}(h^2+k^2)$.

Now the sum of two squares is not divisible by 3, unless each square is divisible by 3, and therefore by 9. Hence $\frac{1}{9}(h^2+k^2)$ is an integer, whence $\frac{1}{3}(m^2+n^2)$ is an integer ; whence m and n are both divisible by 3, which is con-

trary to the hypothesis that m and n have no common factor. Therefore $2m^2+n^2$ and m^2+2n^2 have a common factor. This factor must be a factor also of their sum, *i. e.* $3(m^2+n^2)$.

But m^2+n^2 is prime to $2m^2+n^2$, since if they have a common factor it is also a factor of m^2+n^2 and $(2m^2+n^2)-(m^2+n^2)$, or of m^2+n^2 and m^2, or of n^2 and m^2, which is again contrary to hypothesis.

Therefore the common factor of $2m^2+n^2$ and $m^2+2n^2=3$, and we may put

$$\left.\begin{array}{l} 2m^2+n^2=3h^2 \\ m^2+2n^2=3k^2 \end{array}\right\} \ldots\ldots (7).$$

From the first of these, $2(m^2-h^2)=h^2-n^2$, me may now take $\dfrac{h-n}{2(m-h)}$ in its lowest terms$=\lambda/\mu$, so that

$$2\lambda(m-h)=\mu(h-n), \quad \mu(m+h)=\lambda(h+n),$$

whence $(2\lambda^2+\mu^2)m-(2\lambda^2-\mu^2)h=2\lambda\mu h$, and $m/h=\dfrac{2\lambda\mu+2\lambda^2-\mu^2}{2\lambda^2+\mu^2}$.

Whence as before,

$$\left.\begin{array}{l} h=2\lambda^2+\mu^2 \\ m=2\lambda\mu+2\lambda^2-\mu^2 \\ n=2\lambda\mu-2\lambda^2+\mu^2 \end{array}\right\} \ldots\ldots (8),$$

or the halves of these, respectively, which will have the same form, except that m and n are transposed.

Similarly from the equation $m^2+2n^2=3k^2$, we get

$$m^2-k^2=2(k^2-n^2) \text{ and } 2\lambda_1(k-n)=\mu_1(m-k).$$

Whence
$$\left.\begin{array}{l} k=2\lambda_1^2+\mu_1^2 \\ m=2\lambda_1\mu_1-2\lambda_1^2+\mu_1^2 \\ n=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2 \end{array}\right\} \ldots\ldots (9),$$

or the halves of these, respectively, which will have the same form except that the expressions for m and n will be interchanged.

From the first set $m+n=4\lambda_1\mu_1$, and from the second $m+n=4\lambda_1\mu_1$.

$\therefore\ \lambda\mu=\lambda_1\mu_1$.

Also $2\lambda\mu+2\lambda^2-\mu^2=2\lambda_1\mu_1-2\lambda_1^2+\mu_1^2$,

or $=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2$.

In the first case, $2\lambda^2-\mu^2=-2\lambda_1^2+\mu_1^2$.

$\therefore\ 2(\lambda^2+\lambda_1^2)=\mu^2+\mu_1^2$,

$2\lambda_1^2(\lambda^2+\lambda_1^2)=\lambda_1^2\mu^2+\lambda_1^2\mu_1^2=\lambda_1^2\mu^2+\lambda^2\mu^2=(\lambda^2+\lambda_1^2)\mu^2$.

$\lambda^2+\lambda_1^2$ cannot be zero.

$\therefore\ 2\lambda_1^2=\mu^2$, or $\sqrt{2}\lambda_1=\mu$, which is impossible since λ_1 and μ are integers. We must now therefore try the second case,

$$2\lambda\mu+2\lambda^2-\mu^2=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2.$$

Whence $2(\lambda^2-\lambda_1^2)=\mu^2-\mu_1^2$,

$$2\lambda_1^2(\lambda^2-\lambda_1^2)=\lambda_1^2\mu^2-\lambda_1^2\mu_1^2=\lambda_1^2\mu^2-\lambda^2\mu^2=(\lambda_1^2-\lambda^2)\mu^2.$$

$\therefore\ (2\lambda_1^2+\mu^2)(\lambda^2-\lambda_1^2)=0.$

$2\lambda_1^2+\mu^2$ cannot be zero.

$\therefore$ the only solution is $\lambda^2-\lambda_1^2=0$, or $\lambda=\lambda_1$.

Hence from $\lambda\mu=\lambda_1\mu_1$, $\mu=\mu_1$.

Now $h=2\lambda^2+\mu^2=2\lambda_1^2+\mu_1^2=k$ [from (8) and (9)].

$\therefore\ 2m^2+n^2=m^2+2n^2$ [from (7)].

$\therefore\ m^2=n^2$.

But m and n have by hypothesis no common factor (greater than unity).

$\therefore\ m=n=1$.

$\therefore\ q-p=r-s$ [from (6)], and $pq=rs$.

$\therefore\ q+p=r+s$, and $q=r$, $p=s$. Also $2q-p=2s-r=2p-q$.

$\therefore\ q=p$, and as they have no common measure, each $=1$.

$\therefore\ p=q=r=s=1$, and

$$\left.\begin{aligned}x&=p^2+2q^2-2pq=1\\ y&=4pq-2q^2-p^2=1\end{aligned}\right\}\text{ from (4).}$$

$$\begin{aligned}a&=\sqrt{(2x^2-y^2)}=1\\ b&=\sqrt{(2y^2-x^2)}=1.\end{aligned}$$

(These values of x and y are also derivable from the equations

$$\begin{aligned}x&=4rs-2s^2-r^2\\ y&=2s^2+r^2-2rs,\end{aligned}$$

so that all the equations are consistent.)

$\therefore\ x=y=a=b=1$ is the only solution in integers of the two equations when x and y are prime to one another.

Of course an infinite number of solutions may be obtained by multiplying x, y, a, and b by a common factor, C, so that

$$\begin{aligned}2C^2-C^2&=C^2 \text{ for } 2x^2-y^2=a^2,\\ 2C^2-C^2&=C^2 \text{ for } 2y^2-x^2=b^2.\end{aligned}$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

114. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Does it pay a $4-carpenter using a dozen four-penny nails per minute, to pick up a dropped nail? At this rate, should twenty-penny nails be picked up?

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

Where shall a pole 120 feet high be broken so that the top may rest on the ground 40 feet from the foot? (Solve by arithmetic.)

⁂ Solutions of these problems should be sent to B. F. Finkel not later than August 10.

ALGEBRA.

102. Proposed by J. MARCUS BOORMAN, Woodmere, N. Y.

Solve $2x+\sqrt{(x^2-7)}=5$. [See *Hind's Algebra*, page 447.]

103. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Given the equation $x^m+p_1x^{m-1}+p_2x^{m-2}+\ldots\ldots p_{m-1}x+p_m=0$ freed from multiple roots. Prove that its discriminant is positive or negative according as the number of pairs of complex roots is even or odd.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than August 10.

104. Prize Problem. $2.50 for the best solution.

Compute to three decimal places each of the roots of the equation $x^2+y=2$, $x+y^2=6$.

⁂ Solutions of this problem should be sent to B. F. Finkel not later than September 1.

GEOMETRY.

123. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

A étant le point d' intersection des médianes d' un triangle ABC, démontrerque $AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2)$. [Ex. 84, Géométrie. No. 2, 1re Anne *L' Éducation Mathématique.*]

⁂ Solutions of this problem should be sent to B. F. Finkel not later than August 10.

CALCULUS.

92. Proposed by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.

How much wood is taken from a log 12 inches in diameter, by boring a two-inch hole through the center, the axis of hole being perpendicular to axis of log?

93. Proposed by **JOHN R. JEFFREY**, Student in Ohio State University, Columbus, Ohio.

Solve the following differential equation :

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=2x, \text{ when } x<1.$$

94. Proposed by **ALOIS F. KOVARIK**. Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

Find the minimum isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis. [Ex. 16, page 166, *Rice and Johnson's Differential Calculus*.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

MECHANICS.

91. Proposed by **CHARLES C. CROSS**, Whaleyville, Va.

The bow of a boat which is a inches wide is inclined at an angle α. When in motion in perfectly calm water the water was found to rise b inches on the bow. Required the velocity of the boat.

92. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge. Mass.

A particle, starting at the vertex, slides down a smooth parabolic curve. Prove that in order to leave the curve at the extremity of the latus rectum, the initial velocity of the particle must be $pg[\sqrt{2}-1]$ where p is semi-latus rectum.

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

AVERAGE AND PROBABILITY.

75. Proposed by **F. P. MATZ, M. Sc., Ph. D.**, Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the mean area of all plane rectilineal right triangles having a constant perimeter p.

76. Proposed by **F. P. MATZ, M. Sc., Ph. D.**, Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

In a given ellipse, the extremities of a focal chord are joined with the center. Find the average area of the triangle thus formed.

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

MISCELLANEOUS.

79. Proposed by **S. HART WRIGHT, A. M., Ph. D.**. Penn Yan, N. Y.

In latitude $42^\circ\ 30'$ N.$=\lambda$, a tree 100 feet long$=\alpha$, leans in the direction S. 60° W.$=\beta$, with an angle of elevation with the level ground, of $30^\circ=\gamma$. The sun's declination being $1^\circ\ 36'\ 24''$ N.$=\delta$, in what direction will the shadow of the tree point, when the sun is on the meridian ?

80. **Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.**

Exhibit ten initials in that infinite series of integral, rational rhombuses wherein the area of every term is one unit less than the square of its side.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

EDITORIALS.

It should have been stated in our last issue that the excellent portrait of Sophus Lie was furnished us through the kindness of Dr. Halsted.

The University of Wisconsin is offering for the Academic year beginning 1899, a very excellent course of Graduate Study in Electrical Engineering.

We offer again in this issue a prize of $2.50 for the best solution of problem 104 in algebra. Any person under twenty-one years of age is eligible to compete for the prize.

Advanced sheets of a School Algebra, by Drs. Fisher and Schwatt of the University of Pennsylvania have just reached us. A further notice of the book will be given in the June number of the MONTHLY.

Dr. George Bruce Halsted has been invited to present a Report on Progress in non-Euclidean Geometry at the coming Columbus meeting of the American Association for the Advancement of Science. Dr. Halsted has accepted the invitation and will commence soon the preparation of the report, which is to be quite exhaustive.

During the year 1899-1900, Drury College will offer the following electives in Mathematics: Advanced Integral Calculus, Differential Equations, Projective Geometry, Analytical Mechanics, and Theory of Functions.

The Register and Eleventh Official Announcement of Clark University has just reached us. The following are some of the Courses offered in Mathematics for the year 1899-1900: Differential Geometry, Algebraic Invariants, Analytical Geometry of Higher Surfaces and Twisted Curves, Elliptic Functions, Differential equations, and Calculus of Variations, Finite Continuous Groups, and Theory of Numbers.

The following are some of the advanced courses of Mathematics offered for the year 1899-1900 at the University of Chicago: Twisted Curves and Surfaces, Associate Professor Maschke; Projective Geometry, Professor Moore; Theory of Invariants, Professor Bolza; Continuous Groups, Professor Bolza; Theory of Functions of a Complex Variable, Professor Moore and Associate Professor Maschke; Elliptic Functions, Professor Bolza; Hyperelliptic Functions, Professor Bolza; Abstract Groups, Associate Professor Maschke; Elliptic Modular Functions, Professor Moore; Theory of Substitution, Professor Moore; Theory of Numbers, Assistant Professor Young, etc., etc.

Harvard University has just issued its Course of Study in Mathematics for the year 1899-1900. Among the courses offered in advanced Mathematics are the following: General Theory of Surface, Prof. J. M. Peirce; Dynamics of a Rigid Body, Professor Byerly; Quaternions with Applications to Geometry and Mechanics, Prof. J. M. Peirce; Trigonometric Series, Introduction to Spherical Harmonics, Potential Functions, Professors Byerly and B. O. Peirce; Theory of Functions (Second Course), Riemann's Theory of Functions, Professor Osgood; Algebra—Galois's Theory of Equations, Professor Osgood; Lie's Theories as Applied to Differential Equations, Dr. Bouton; etc., etc. The courses offered at Harvard are sufficiently varied and extensive to meet the wants of any student of mathematics.

BOOKS AND PERIODICALS.

A Text-Book of Physics.—Sound. By J. H. Poynting, Sc. D., F. R. S., Late Fellow of Trinity College, Cambridge, Professor of Physics in Mason University College, Birmingham; and J. J. Thomson, M. A., F. R. S., Sc. D., Dublin; Hon. D. L., Princeton; Fellow of Trinity College, Cambridge; Cavendish Professor of Experimental Physics in the University of Cambridge. Royal 8vo. Cloth, 174 pages. Price 8*s*. 6*d*. London: Charles Griffin & Co., and Philadelphia: J. B. Lippincott & Co.

The authors are preparing a Text-book of Physics of which the account of the phenomena of sound and the the theory of these phenomena as presented in this book forms only one part. The presentation of the subject of sound as here given is such as to enable the student who has no knowledge of advanced mathematics to obtain a good understanding of this branch of physics. The book is well printed, the quality of paper used is very good, but the illustrations of some pieces of apparatus are poor. B. F. F.

The Elements of Graphic Statics. A Text-book for Students of Engineering. By L. M. Hoskins, Professor of Applied Mechanics in the Leland Stanford Junior University. Revised Edition. 8vo. Cloth, 200 pages. Price, $2.25. New York: The Macmillan Co.

In this book is found not only that which meets the needs of the student, but also that which is useful to the structural engineer as well. B. F. F.

Röntgen Rays. Memoirs by Röntgen, Stokes, and J. J. Thomson. Translated and edited by George F. Barker, LL. D., Professor of Physics in the University of Pennsylvania. 8vo. Cloth, 76 pages. Price, 60 cents. New York: Harper & Bros.

This is volume III of Harper's Scientific Memoirs, edited by J. S. Ames, Ph. D., Professor of Physics in Johns Hopkins University. This little book contains Professor Röntgen's paper in which he announced to the world his wonderful discovery; a biography of Professor Röntgen; a paper On the Nature of the Röntgen Rays, by Sir G. G. Stokes; a biographical sketch of Stokes; a paper On the Theory of the Connection between Cathode and Röntgen Rays, by J. J. Thomson; and a biographical sketch of Thomson. This little

volume makes very interesting reading for any one interested in one of the most wonderful discoveries of the nineteenth century. B F. F.

A History of Physics in its Elementary Branches, including the Evolution of Physical Laboratories. By Florian Cajori, Ph. D., Professor of Physics in Colorado College. 8vo. Cloth, 322 pages. Price, $1.60. New York: The Macmillan Co.

This History of Physics is in line of excellence and interest with the History of Mathematics by Dr. Cajori. Such an account of the progress and development of Physics, as well as a short sketch of the investigators who have enlarged and enriched its domain by careful research and discovery, has long been found desirable. Every teacher and student of Physics should read this book, as the history of any science has a stimulating effect and helps to make it attractive. De Morgan said, "No man should think it a waste of time to learn something of the history of his own subject; nor is the investigation of laborious methods now fallen into disuse, or errors once commonly accepted the least valuable of mental disciplines." B. F. F.

The Modern Theory of Solution. Memoirs by Pfeffer, Van't Hoff, Arrhenius, and Raoult. Translated and edited by Harry C. Jones, Ph. D., Associate in Physical Chemistry in Johns Hopkins University. 8vo. Cloth, 128 pages. Price, $1.00. New York: Harper & Bros.

This is the fourth volume of Harper's Scientific Memoirs. It contains Pfeffer's Osmotic Investigations; biography of Pfeffer; Van't Hoff's The Role of Osmotic Pressure in the Analogy between Solutions and Gases; biography of Van't Hoff; Arrhenics's On the Dissociation of Substances Dissolved in Water; biography of Arrhenius; Raoult's The General Law of the Freezing of Solvent and On the Vapor-Pressure of Etheriel Solutions; The General Law of the Vapor-Pressure of Solvents; and a biography of Raoult. B. F. F.

Algebra Elementare Ad Uso Dei Licei e Degli Instituti Fecnici (I° Biennio) Secondo I Programmi Governativi con Copiose Note Storiche Molte Consigli Pratici Per Indirizzare L' Allunno Alla Resoluzione Degli Esercizi Piu' di 2000 Esercizi e Problemi Graduati da Risolvere e Circa 400 Esercizi e Problemi Minutaments Risolti. Pages 428.

This is an elementary algebra of unusual value, as the definition of all algebraic operations are given according to the advanced notions of modern mathematics. The numerous illustrative problems are worked out in detail, the historical notes are of value, and the many problems of the various chapters well chosen. The work throughout bears evidence of the painstaking care of the author in its preparation. B. F. F.

The following periodicals have been received: *Journal de Mathématiques Élémentaires*, (1er Mai 1899); *The American Journal of Mathematics*, (April, 1899); *L'Intermédiaire des Mathématiciens*, (April 1899); *Bulletin of the American Mathematical Society*, (May, 1899); *The Kansas University Quarterly*, (January, 1899); *The Monist*, (April, 1899): *The Ohio Teacher*, (April, 1899); *The Educational Times*, (May 1, 1899); *The Mathematical Gazette*, (February, 1899).

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1 00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

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ON SYMMETRIC FUNCTIONS.

By E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Concluded from May Number.]

H. APPLICATION OF THE THEORY TO COMPUTATION.

The limits of the present article do not permit of thoroughgoing application in the calculation of coefficients of terms of symmetric functions. The writer has used the present theory and the formula of G, 2, in such calculations, and by means of the parallel theory for the resultant, and the equivalent formula of A, 4, he has calculated all the coefficients of the normal forms of all resultants up to and including the resultant $R_{5,\,4}$.* By means of these methods the calculation is very easy. The advantages of the theory and formulas here developed in the calculation of tables of symmetric functions may be stated to be as follows: Not only is the symmetry of the table established by the fundamental reiations, whereby half the coefficients are repeated, but also by the same relations the numerical equality of certain other coefficients in the same table and of others in different tables is immediately established. In addition to this many coefficients of a table which are not normal forms are easily reduced to such as are normal forms of a lower table and have been previously calculated. Also the coefficients of the completely reducible forms, which have the form of the general formula obtained for them in F, 2, are the values of $(-1)^{w}$. To this may be added certain coefficients whose value is zero by the fundamental conditions. [We might

*The results are published in the before-mentioned thesis.

also add the general formula, for the normal forms where $n=2$, which can be proved without difficulty, viz :

$$\binom{0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}}{0^m\quad m\quad 0}=(-1)^{\lambda_1+\lambda_2}\frac{m(\lambda_1+\lambda_2-1)!}{\lambda_1!\ \lambda_2!}].$$

When all these means of obtaining coefficients have been applied, the actual number of normal forms in a table requiring calculation is comparatively very small, and by means of the formula of F, 2, the calculation is easily made, giving the coefficient at once as the sum of earlier calculated coefficients.

NOTE ON ELIMINATION BY MEANS OF SYMMETRIC FUNCTIONS.

In behalf of the extension of the method given under I, B, the following details may be added :

1. On the Resolution of Aronhold's Operator into Three Operators.

The term containing $(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$ in $\delta R_{m,n}$ must have come from $a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}\mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid$ by the use of $b_iD_{a_i}$ in as many ways as there are operators of this kind when i takes all values from $i=0$, to $i=n$. $b_iD_{a_i}$ applied to $a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}\mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid$ gives

$$(\lambda_i+1)b_i\mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$$

$$=(\lambda_i+1)b_i\mid 0^{\lambda_0}1^{\lambda_1}\ldots.i^{\lambda_i+1}\ldots.m^{\lambda_m}\mid(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m},$$

and the coefficient of $(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$ in $\delta R_{m,n}$ is

$$\sum_{i=0}^{i=n}(\lambda_i+1)b_i\mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid\equiv 0.$$

Thus Aronhold's operator is resolved into three operators :

(1). $0, 1, 2, \ldots.n$ *applied to* $\mid 0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid$.

(2). *The literal operators* $b_0, b_1, \ldots\ b_n$ *applied to the preceding.*

(3). *The numerical operators* $\lambda_0+1, \lambda_1+1, \ldots.\lambda_n+n$.

Here λ_i+1 associated with b_i is the exponent of i in the associated stroked form which results from the first operation. Of course $\lambda_0+\lambda_1+\ldots.\lambda_m=n-1$, and there will be

$$\frac{m(m+1)\ldots.(m+n-1)}{1.2\ldots.(n-1)}=e,$$

(the number of homogeneous products of m elements to $n-1$ dimensions) such identical equations of the form

$$\sum_{i=0}^{i=n} (\lambda_i+1)b_i \mid i0^{\lambda_0}1^{\lambda_1}\ldots . m^{\lambda_m} \mid \equiv 0.$$

The reader will observe the identity of this equation with the one from which the recurrence formula for the normal forms of symmetric functions was derived.

2. On the Number of Functions Requiring Calculation in $R_{m,n}$, and the Sufficiency of Aronhold's Operator.

It is evident that every symmetric function of the form

$$b_0{}^m \Sigma(\beta_1)^{\kappa_1}(\beta_2)^{\kappa_2}\ldots . (\beta_n)^{\kappa_n},$$

in $R_{m,n}$, where $\kappa_1 \geqq \kappa_2 \geqq \ldots . \geqq \kappa_n > 0$, can be reduced to the form

$$(-1)^{n\kappa_n}(b_n)^{\kappa_n}(b_0)^{m-\kappa_n} \Sigma (\beta_1)^{\kappa_1-\kappa_n}(\beta_2)^{\kappa_2-\kappa_n}\ldots . (\beta_{n-1})^{\kappa_{n-1}\kappa_n}.$$

Hence only those functions require calculation which contain in any term less than all the n roots; and of these we know all the fundamental symmetric functions,

$$b_0 \Sigma\beta_1, \quad b_0 \Sigma\beta_1\beta_2, \quad \ldots . \quad b_0 \Sigma\beta_1\beta_2 \ldots . \beta_{n-1}.$$

Let N denote the number of functions requiring calculation. All the terms in $R_{m,n}$ in which a_m is not a factor, contain functions in which n roots enter, and are therefore of the before mentioned reducible form, and are dependent upon such forms as do not contain n roots in a term. These latter irreducible (in this sense) forms are all found among the terms of $R_{m,n}$ which contain a power of a_m, and among these are none which contain n roots in a term. We may note that the terms of the form $(b_0)^m(a_m)^{m-r}(a_{m-1})^r \Sigma\beta_1\beta_2 \ldots . \beta_r$ from $r=0$, to $r=n-1$, n in number are found among them and are known. We have $N+n=$ the number of terms in $R_{m,n}$ containing a_m. The whole number of terms or functions, f, in the resultant $R_{m,n}$ is

$$f=\frac{(m+1)(m+2)\ldots .(m+n)}{1.2\ldots .n}.$$

The number of them not containing a_m could be found by putting $a_m=0$. It is the same as the number of terms in $R_{m-1,n}$, and is

$$\frac{m(m+1)\ldots .(m+n-1)}{1.2\ldots .n}.$$

Therefore,

$$N+n=\frac{(m+1)(m+2)\ldots .(m+n)}{1.2\ldots .n}-\frac{m(m+1)\ldots .(m+n-1)}{1.2\ldots .n}=f-\frac{m}{n}e$$

$$=\frac{(m+1)(m+2)\ldots.(m+n-1)}{1.2\ldots.(n-1)}=e, \text{ or}$$

$$N+n=f-\frac{m}{n}e=e, \text{ and } N+n=e=\frac{n}{m+n}f\leqq\tfrac{1}{2}f,$$

and we see:

(1). *The number of identical linear equations furnished by Aronhold's operator*, and containing the N unknown functions together with n others, *is just equal to the number $N+n$, and therefore jast sufficient to compute both the unknown N as well as also the n other functions*, if we regard the latter as unknown. This proves the sufficiency of Aronhold's operator for calculating the resultant by means of symmetric functions.

(2). *Certainly less than one-half of the whole number of symmetric functions which enter into the resultant require calculation.*

$$\begin{aligned}\text{If } m=n,\ & N+n=\tfrac{1}{2}f,\\ n=1,\ & N=0,\ e=1,\ f=2,\\ n=2,\ & N=1,\ e=3,\ f=6,\\ n=3,\ & N=7,\ e=10,\ f=20,\ \text{etc.},\end{aligned}$$

which agrees with the fact that we found 7 functions requiring calculation in $R_{3,\,3}$.

3. Farther Remarks on Calculating $R_{m,n}$.

(1). *Recurrence methods.*

Since the sum of all the terms which do not contain a_m is equal to $b_n R_{m-1,\,n}$, we have

$$R_{m,n}=b_n R_{m-1,\,n}+\Sigma\lambda\sum_{i=0}^{i=n} a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}\mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m}\mid$$

where the exponent of a_m must not be zero, and where $\lambda_0+\lambda_1+\ldots.\lambda_m=n-1$. The second portion of the right hand expression contains all the unknown functions while the first term is a previously calculated resultant. By giving m and n special values, or requiring them to satisfy certain relations, like $m=n+1$, etc., various recurrence formulas may be obtained and used.

(2). *Direct calculation of $R_{m,n}$.*

If one does not choose to proceed by recurrence formulas we have shown that Aronhold's operator furnishes a sufficient number of equations that are written down by an easy rule, to calculate $R_{m,m}$ directly and independently. From $R_{m,m}$ we can obtain $R_{m,n}$ where $n=m-r$, at once by the formula

$$R_{m,m-r}=(-1)^{mr}\left(\frac{R_{m,m}}{a_m{}^r}\right)_{b_m=b_{m-1}=\ldots.b_{m-r+1}=0}.$$

To this may be added the relation, $R_{m,n}=(-1)^{mn}R_{n,m}$.

BIBLIOGRAPHICAL REFERENCES.

In the preparation of this paper many of the fundamentals of Algebra have been assumed as being familiar. For the benefit of any readers who may feel the need of them, the following references are given to cover the ground which has thus been assumed. The references are intended in no sense to form a complete bibliography.

In the theorems concerning order and weight of symmetric functions, see Faà di Bruno, Einleitung in die Theorie der Binären Formen, German Translation by Dr. Theodor Walter, Leipzig, B. G. Teubner, 1881. s. TU. 13; also Burnside and Panton, Theory of Equations, second edition, London: Longmans, Green & Co., 1886, pp. 299 *et seq.*

On corresponding and conjugate forms of symmetric functions, see Faà di Bruno, loc. cit. Anhang, s. 302 *et seq.*

For tables of symmetric functions. Ibid. s. 311 *et seq.*

For a pretty complete bibliography of the literature up to 1881, of symmetric functions, resultants, and related subjects, see Faà di Bruno, Anhang. s. 373 *et seq.*

On the resultant of two binary forms and its elementary properties, see Faà di Bruno, loc. cit. §5. Ueber die Bildung der Resultanten. s. 50 *et seq.* §6. Eigenschaften der Resultanten, s. 75 *et seq*; also Gordan, Invariantentheorie, Erster Band, Leipzig, B. G. Teubner, 1885. §§ 10 u. 11, s. 145 *et seq*; also Weber, Lehrbuch der Algebra, Zweite Auflage, Erster Band, Braunschweig, F. Vieweg und Sohn, 1898. §53, Resultanten, s. 175 *et seq*; also Burnside and Panton, loc. cit. Chapter XIII, pp. 318 *et seq.*

On the resultant in terms of symmetric functions, see the references already given.

In addition to the preceding, the works of Salmon either in German or in English may be consulted.

On substitutions and their application to determinants, so far as employed in this paper, see Gordan, loc. cit., s. 1—31.

On Aronhold's operator, see Gordan, Invariantentheorie, Zweiter Band, Leipzig, B. G. Teubner, 1887. s. 60 *et seq.*

For proof of the proposition that Aronhold's operator applied to the resultant gives identically zero, or that $\delta R_{m,n}\equiv 0$, see Faà di Bruno, loc. cit. s. 79, Anmerkung 3.

On corresponding matrices and corresponding determinants, see Gordan, Invariantentheorie, Erster Band, s. 94 *et seq.*

On the number of homogeneous products of n elements to r dimensions, see Todhunter, Algebra, London: Macmillan & Co., 1881. p. 316.

URKUNDEN ZUR GESCHICHTE DER NICHTEUKLIDISCHEN GEOMETRIE VON F. ENGEL UND P. STAECKEL.

By GEORGE BRUCE HALSTED, University of Texas, Austin, Texas.

I. *Nikolai Ivanovitsch Lobatschefski.* Leipzig. B. G. Teubner. 1899. 8vo. pp. 476.

The name of Lobachévski is inseperably connected with a scientific advance so fundamental as actually to have changed the accepted conception of the universe.

Yet his first published work and his greatest work have both remained for over sixty years inaccessible, locked up in Russian, and are now for the first time given to the world in this monumental volume by Professor Engel.

As to the precise time at which Lobachévski shook himself free from Euclid's two thousand years of authority there is still room for a most interesting doubt.

The first of the two treatises given in this book, "On the Elements of Geometry," was published in 1829, with this note at the foot of the first page:

"Extracted by the author himself from a paper which he read February 12, 1826 in the meeting of the Section for physico-mathematic sciences, with the title 'Exposition succincte des principes de la Géometrie etc.' "

Again, when the four equations are reached which really contain the essence of the non-Euclidean Geometry, Lobachévski subjoins this note: "The equations (17) and all that follows these the author had already appended to the paper which he presented in 1826 to the Section for physico-mathematic sciences."

In the introduction to the second of the two treatises here given, the "New Elements of Geometry," the author says, "Everyone knows, that in geometry the theory of parallels has remained, even to the present day, incomplete.

"The futility of the efforts which have been made since Euclid's time during the lapse of two thousand years to perfect it awoke in me the suspicion that the ideas employed might not contain the truth sought to be demonstrated, and for whose verification, as with other natural laws, only experiments could serve, as for example, astronomic observations.

"When finally I had convinced myself of the correctness of my supposition, and believed myself to have completely solved the difficult question, I wrote a paper on it in the year 1826: Exposition succincte des principes de la Géométrie, *avec une démonstration rigoureuse du théorème des parallèles*, read February 12, 1826, in the séance of the physico-mathematic Faculty of the University of Kazan, but never printed."

No part of this French manuscript has ever been found. The latter half of the title is ominous.

For centuries the world has been deluged with rigorous demonstrations of

the theorem of parallels. We know that three years later Lobachévski himself proved it absolutely indemonstrable. Yet the paper said to contain material to stop forever this twenty-centuries-old striving still was headed 'démonstration rigoureuse,' just as Saccheri's book of 1733 containing a coherent treatise on non-Euclidean geometry ended by one more pitiful proof of the parallel postulate.

If Saccheri had lived three years longer and realized the pearl in his net, with the new meaning he could have retained his old title 'Euclides ab omni naevo vindicatus,' since the non-Euclidean geometry is a perfect vindication and explanation of Euclid. But Lobachévski's title is made wholly indefensible. A new geometry founded on the contradictory opposite of the theorem of parallels and so proving every demonstration of that theorem fallacious, could not very well pose under Lobachévski's old title. Least said, soonest mended. He never tells what he meant by it, never tries to explain it. Yet Engel thinks that under this two-thousand-years stale title "avec une démonstration rigoureuse du théorème des parallèles," "Lobatschefskij sprach es klipp und klar aus, dass das Euklidische Parallelenaxiom niemals werde bewiesen werden koennen, weil es unbeweisbar sei."

At the international mathematical congress, 1893, I maintained in his presence that Felix Klein was utterly in error where in his Nicht-Euklidische Geometrie, I. p. 174, he says of the letter from Gauss to Bolyai Farkas, 1799, "In this last letter is particularly said, that in the hyperbolic geometry there is a maximum for triangle-area"; aud again where he says, p. 175, "There can be no doubt that Lobachévski as well as Bolyai owe to Gauss's prompting the initiative of their researches."

Klein's only answer was that his position would be sustained when the the public got access to Gauss's correspondence. Staeckel and Engel have now had complete access to these papers, and this is what Engel says, pp. 428, 9: "But at all events in Gauss's letters there is nowhere a support for this tradition ; at no point of these letters can be found even the slightest intimation that Gauss connected the discoveries of Lobachévski and J. Bolyai with any direct or roundabout prompting from him. On the contrary the letters show (see p. 432 f. and Math. Ann. 49, p. 162, Briefwechsel G. B. p. 109), that Gauss throughout recognized the independence of both, exactly as he recognized that of Schweikart whose independence of Gauss is subject to no doubt. With Staeckel I am at one herein, that exactly this circumstance is particularly weighty for the decision of the whole question."

The whole scientific world will breath a sigh of relief that Klein's Goettingen legend, wounded in 1893, is in 1899 annihilated forever. More inexplicable is Klein's misinterpretation of Gauss's letter of 1799 to Bolyai Farkas.

I gave this letter in my Bolyai as demonstrative evidence that in 1799 Gauss was still trying to prove Euclid's the only non-contradictory system of geometry, and also the system regnant in the external space of our physical experience.

The first is false ; the second can never be proven.

Summing up this same letter, Engel, p. 379, instead of finding in it the hypothetical white elephant of Klein's fairy tale, gives the utmost that can be attributed to it in the following sentence : "Hier ist er also ganz nahe daran, an der Richtigkeit der Geometrie, das heisst, des Euklidischen Parallelenaxioms zweifelhaft zu werden."

Five years later, in a letter of November 25, 1804, Gauss speaks of a "group of rocks" on which his attempts had always been wrecked, and adds, "I have indeed still ever the hope that those rocks sometime, and indeed before my death, will permit a passage. Meanwhile I have now so many other affairs on hand, that at present I cannot think on it, and believe me, I shall heartily rejoice if you forestall me, and if you succeed in surmounting all obstacles." "Surely," says Engel, "that does not sound as if the authority of Euclid had diminished in power since the year 1799 ; on the contrary, one gets the impression that Gauss in 1804 rather stood more completely under its ban than before."

This was clearly the view of Bolyai János, whose autobiography, after quoting Gauss's letter of 1832, says : "In a previous letter Gauss writes : He hopes sometime to be able to circumnavigate these rocks—so then he hopes ! !" "These last words," say Staeckel and Engel in the Mathematische Annalen, "show a certain suspicion on the part of John against Gauss." But the mention of this earlier letter was highly natural. János had known of it from boyhood. The joy of his triumph in solving what had baffled all the world for two thousand years was intensified by his knowing that even Gauss had tried and was hoping the impossible.

His splendid trumpet call of glory announcing his creation of a new universe, scientiam spatii absolute veram exhibens, is answered how ? Gauss answers; that method and results coincide with his own *meditations* instituted in part since 30—35 years. But of these meditations Gauss had published never a word ! How natural then for János to refer to his previous letter where he still was hoping to prove Euclid's parallel postulate.

The equally complete freedom of Lobachévski from the slightest idea that Gauss had ever meditated anything different from the rest of the world on the matter of parallels is demonstrated most happily.

Bartels, the teacher of Lobachévski, never saw Gauss after 1807, received at Kazan one letter from him in 1808, probably a mere friendly epistle containing nothing mathematical, and not another word during his entire stay there.

But in November, 1808, Schumacher in Goettingen writes in his dairy that Gauss has reduced the theory of parallels to this, that if the accepted theory were not true there must be a constant *a priori* of length, "welches absurd ist." Yet that Gauss himself considers this work not yet completed.

Thus in 1808 Gauss still vacillates. The proposition about the *a priori* given unit for length is due to Lambert 1766, and on the supposed absurdity Legendre in 1794 had founded a pseudo-proof of the parallel postulate.

Thus until after 1808 Gauss had made no advance beyond the ordinary text-books.

A most fortunate piece of personal testimony from the distinguished astronomer Otto Struve finishes the whole matter.

When at Dorpat in 1835 and 1836 Struve was attending his lectures, Bartels repeatedly spoke of Lobachévski as one of his first and most gifted scholars in Kazan. Lobachévski had then already sent his first works on non-Euclidean geometry to Bartels, but, as Struve writes, Bartels looked upon these works "more as interesting, ingenious speculations, than as a work advancing science." Struve adds, he does not recall that Bartels ever spoke of any accordant ideas of Gauss.

Such misconception of the import of non-Euclidean geometry was due in part to that lack of grit or slip in judgment which let Lobachévski damn this child of his genius with the name "Imaginary Geometry."

If Lobachévski had possessed the magnificent Magyar mettle of Bolyai János, and dared to name his creation the Science Absolute of Space, he would not have taught mathematics with ability throughout his life without making a single disciple.

His New Elements of Geometry, here at last made accessible to the world, is such a masterpriece that it remains to-day the completest and most satisfactory text book of non-Euclidean geometry. Written at the flood of hope and confidence, with ardor still undampened, it is in his 'New Elements' preëminently that the great Russian allows free expression to his profound philosophic insight, which on the one hand shatters forever Kant's doctrine of our absolute *a priori* knowledge of all fundamental spatial properties, while on the other hand emphasizing the essential relativity of space, and the element of human construction, human creation in it.

Lobachévski's position is still after sixty years the necessary philosophy for science. No one has succeeded in finding any escape from its cogency. No one has gone beyond it.

Our hereditary geometry, the Euclidean, is underivable from real experience alone, and can never be proved by experience. Not only can the truth or falsity of Euclid's parallel postulate never be proved *a priori;* not even *a posteriori* can ever its *truth* be proved.

Therefore Euclidean geometry, in so far as Euclidean, must ever remain a creation of the human mind.

The introduction to the New Elements contains a piercing critique of Legendre's attempts on the parallel postulate. Here at times Lobachévski almost condescends to be humorous. For example, he says: "Although Legendre designates his demonstration as completely rigorous, he without doubt thought otherwise, for he adds the proviso, that a difficulty which one would perhaps still find can always be removed. For this, he has recourse to calculations founded on the first familiar equations of rectilinear trigonometry, which it would be necessary previously to establish, and which just in this case are useless and lead to no result."

Here for the word *Trigonometry* in the Russian of the Collected Works, p. 222, Engel has substituted, p. 70, by some slip, the word *geometry*. Further on

Lobachévski continues: "but Legendre has not noticed here, that EF may possibly not meet AC. To overcome *this little difficulty*, you have only to suppose that EF is the perpendicular from F on BD; but then how can we conclude therefrom that $FE=AB$ and the angle $EFC=\frac{1}{2}\pi$? It is not possible to mend the false deduction, wherein Legendre's inadvertence was so gross that without remarking this grave error, he considered his demonstration as very simple and perfectly rigorous."

Now for a specimen of Lobachévski's philosophizing. "Strictly we cognize in nature only motion, without which sense-impressions are not possible. Consequently all other ideas, for example, geometric, are artificial products of our mind, since they are taken from the properties of motion; and therefore space in itself, for itself alone, for us does not exist. Accordingly it can have nothing contradictory for our mind, if we admit that some forces in nature follow the one, others another special geometry.

To illustrate this thought, assume, as many believe, that the attractive forces diminish because their action spreads on a sphere. In the ordinary geometry we find $4\pi r^2$ as magnitude of a sphere of radius r, whence the force must diminish in the squared ratio of the distance.

In the imaginary (sic) geometry I have found the surface of the sphere equal to $\pi(e^r-e^{-r})^2$, and possibly such a geometry the molecular forces may follow, whose whole diversity would depend, consequently, on the number e, always very great."

How far Lobochévski was not only from Riemann's geometry, with closed finite straight line, but also from the perspective point of view where the straight is closed by having only one point at infinity, is illustrated by the following sentences of the introduction: "I consider it not necessary to analyze in detail other assumptions, too artificial or too arbitrary. Only one of them yet merits some attention; the passing over of the circle into a straight line. However, the fault is here visible beforehand in the violation of continuity, when a curve which does not cease to be closed howsoever great it may be, transforms itself directly into the infinite straight, losing in this way an essential property. In this regard the imaginary geometry fills in the interval much better, In it if we increase a circle all of whose diameters come together at a point, we finally attain to a line such that its normals approach each other indefinitely, even though they can no longer cut one another. This property, however, does not appertain to the straight but to the curve which in my paper "On the Elements of Geometry" I have designated as *circle-limit*."

Lobachévski anticipated in 1835 all that was said not long ago in the columns of *Science* on the length of a curve. For example: "In fact, however little may be the parts of a curve, they do not cease to be curves; consequently they can never be measured by the aid of a straight. Lagrange takes as foundation the assumption of Archimedes that on a curve one can always take two points so near that the arc between them may be considered greater than its chord but smaller than the two tangents from its extremities. Such an as-

sumption is actually necessary, but by it is destroyed the primitive idea of measuring curve with straights.

Thus the evaluation of the length of a curve represents not at all the rectification of the curvature ; but it seeks a wholly different aim : the finding of a limit which the actual measure would approach the more as this measure was made the more exact. But measuring is considered more exact the smaller the links of the chain employed. This is why in geometry one must show that the sum of tangents decreases while the sum of chords increases until the two sums differ indefinitely little from the limit both approach, which geometry assumes as length of the curve."

In the splendid treatise which follows this interesting introduction Lobachévski has given a complete coherent development and exposition of the non-Euclidean geometry.

Until I visited Maros-Vásárhely it was not known that Bolyai János had actually commenced and made remarkable progress in an even greater, more masterful treatment of the whole matter.

From the mass of John's papers tumbled in a big chest, I singled out especially a manuscript in German entitled "Raumlehre," and on pointing out to Professor Bedöházi János some of the striking passages in it, he promised its publication.

In *Science* for September 24, 1897, I mentioned these treasures as "extended researches anticipating the discoveries of Cayley and Klein." Engel now says of them, p. 393 : "J. Bolyai had also commenced to work out a great and consecutive presentation of geometry, but what he had written down remained entombed in his papers and has never been published. Staeckel will before long make generally accessible so much of it as is suitable for publication, and it will then appear, that J. Bolyai in his exposition set to work according to principles similar to those Lobachévski actually followed." But though Lobachévski has given his complete message to the ages, yet is perceptible a touch more masterful in even the brief two dozen pages of the young Magyar.

Through a given point to draw a parallel to a given straight; to draw to one side of an acute angle the perpendicular parallel to the other side; to square the circle ; these problems would be sought in vain in the two quarto volumes of Lobachévski.

Bolyai János gives solutions of them startling in their elegance. For example, (Halsted's Bolyai, §34), "Through D we may draw DM parallel to AN in the following manner. From D drop DB perpendicular to AN; from any point A of the straight AB erect AC perpendicular to AN (in DBA), and let fall DC perpendicular to AC. A quadrant described from the center A in BAC, with a radius$=DC$, will have a point B or O in common with ray BD. In the first case the angle of parallelism manifestly is right ; but in the second case it equals AOB. If therefore we make $BDM=AOB$, then DM will be parallel to BN."

About 100 pages of Engel's book are devoted to a life of Lobachévski, yet no word is said of his wife, his children, his family life, his home fortunes and

misfortunes, nor is mentioned the biography by E. F. Letvenov (St. Petersburg, 1894, p. 79) containing romantic pictures of these eternal interests.

Austin, Texas.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

110. Proposed by **F. P. MATZ, M. Sc., Ph. D.**, Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

By measuring with a yard $m=12\frac{1}{2}\%$ *too short*, my profits are $n=25\%$ of my sales. If my yard be $p=10\%$ *too long*, what per cent. of my sales will be my profits?

I. Solution by **J. K. ELLWOOD, A. M.**, Principal of Colfax School, Pittsburg, Pa., and **B. F. SINE**, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.

$(1-m)$ yards cost $\frac{3}{4}$ of sale price.

$(1+p)$ yards cost $(1+p)\times\dfrac{1}{1-m}\times\frac{3}{4}=\dfrac{3(1+p)}{4(1-m)}$.

$1-\dfrac{3(1+p)}{4(1-m)}=1-\dfrac{3\times1.10}{4\times.87\frac{1}{2}}=\frac{2}{35}$ of sales, the profit.

II. Solution by **JOHN F. TRAVIS**, Student in Ohio State University, Columbus, Ohio.

1. 100%=the sales. Then
2. 75%=cost price of the part sold in terms of the selling price.
3. $\left\{\begin{array}{l} 1.\ 100\%=\text{correct length of a yard.} \\ 2.\ 87\frac{1}{2}\%=\text{length used.} \\ 3.\ 87\frac{1}{2}\%=75\%\ \text{the cost of this part in terms of its selling price.} \\ 4.\ 1\%=\dfrac{1}{87\frac{1}{2}}\ \text{of}\ 75\%=\dfrac{75}{87\frac{1}{2}}\%,\ \text{and} \\ 5.\ 100\%=100\ \text{times}\ \dfrac{75}{87\frac{1}{2}}\%=85\frac{5}{7}\%. \end{array}\right.$

$\therefore$ The cost of the part sold in terms of its selling price, is $85\frac{5}{7}\%$.

4. $\left\{\begin{array}{l} 1.\ 100\%=85\frac{5}{7}\%. \\ 2.\ 1\%=\frac{1}{100}\ \text{of}\ 85\frac{5}{7}\%=.85\frac{5}{7}\%,\ \text{and} \\ 3.\ 10\%=10\ \text{times}\ .85\frac{5}{7}\%=8.5\frac{5}{7}\%. \end{array}\right.$

5. $85\frac{5}{7}\%+8.5\frac{5}{7}\%=94.2\frac{6}{7}\%$, cost of a yard in terms of its selling price, when the measure is 10% too long.

6. $100\%-94.2\frac{6}{7}\%=5\frac{5}{7}\%$ gain.

$\therefore$ In the latter case my gain will be $5\frac{5}{7}\%$ of the selling price.

Also solved by *COOPER D. SCHMITT*, *ALOIS F. KOVARIK*, and *P, H. PHILBRICK*.

111. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

By what per cent. of its original dimensions must a linear yard of steel rail, weighing 60 pounds, be increased so that it may weigh 75 pounds?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.; SYLVESTER ROBINS, North Branch, N. J., and ELMER SCHUYLER, Annapolis, Md.

The dimensions of the enlarged rail would be $(\frac{75}{60})^{\frac{1}{3}}=(\frac{10}{8})^{\frac{1}{3}}=\frac{2\cdot 1544}{2}=$ 1.0772 times those of the original rail, or an increase of 7.72 per cent.

Also solved by *G. B. M. ZERR*, *B. F. SINE*, and *J. F. TRAVIS*.

ALGEBRA.

93. Proposed by ELMER SCHUYLER, High Bridge, N. J.

Given $x^2-yz=1$; $y^2-xz=2$; $z^2-xy=3$. Find x, y, and z.

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Subtracting the first equation from the second, the second from the third, and factoring, we get $x+y+z=\dfrac{1}{y-x}$ and $x+y+z=\dfrac{1}{z-y}$, respectively; whence $y=\dfrac{x+z}{2}$. Substituting this value of y in the equations we get from the first and third $z^2-x^2=\frac{4}{3}$ and from the second $(x-z)^2=8$. From these two equations x and z are easily found to be $\mp\frac{5}{6}\sqrt{2}$ and $\pm\frac{7}{6}\sqrt{2}$, respectively; whence $y=\pm\frac{1}{6}\sqrt{2}$.

II. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

Squaring the first and subtracting the product of the other two, we have

$$x(x^3+y^3+z^3-3xyz)=-5.$$

$$\text{Similarly, } y(x^3+y^3+z^3-3xyz)=1.$$

$$z(x^3+y^3+z^3-3xyz)=7.$$

$$\therefore \frac{x}{-5}=\frac{y}{1}=\frac{z}{7}=\pm\sqrt{\left(\frac{x^2-yz}{25-7}\right)}=\pm\sqrt{\tfrac{1}{18}}=\pm\tfrac{1}{6}\sqrt{2}.$$

i. e. $x=\mp\frac{5}{6}\sqrt{2}$, $y=\pm\frac{1}{6}\sqrt{2}$, $z=\pm\frac{7}{6}\sqrt{2}$.
[See Charles Smith's *Treatise on Algebra*, page 167].

III. Solution by A. H. BELL, Hillsboro, Ill., and J. SCHEFFER, A. M., Hagerstown, Md.

$(1)+(2)+(3)=x^2+y^2+z^2-(xy+xz+yz)=6$ . . . . (4).

$(4)^2$ $(x^2+y^2+z^2)^2-(xy+xz+yz)^2-12(xy+xz+yz)=36$ . . . . . . (5).

$(1)^2+(2)^2+(3)^2$, $(x^2+y^2+z^2)^2-(xy+xz+yz)^2=14$ . . . . . (6).

$(6)-(5)$, $(x+y+z)(xy+xz+yz)=-\frac{11}{6}$ . . . . . . (7).

$3\times(7)+(4)$, $(x+y+z)^2=\frac{1}{2}$ or $x+y+z=\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{2}$ . . . . . . (8).

$2\times(7)+(4)$, $x^2+y^2+z^2+xy+xz+yz=\frac{7}{3}$......(9).

$(1)-(2)-(3)$, $x^2-y^2-z^2+xy+xz-yz=-4$..... (10).

$(9)+(10)$, $x(x+y+z)=-\frac{5}{6}$ by (8) $x=\mp\frac{5}{6}\sqrt{2}$......(11).

$(2)-(1)-(3)+(9)$ etc., $y=\pm\frac{1}{6}\sqrt{2}$......(12).

$(3)-(1)-(2)+(9)$ etc., $z=\pm\frac{7}{6}\sqrt{2}$......(13).

IV. Solution by G. I. HOPKINS, Manchester, N. H.

Subtract (1) from (2) and (3).

$y^2-x^2+yz-xz=1$......(4).

$z^2-x^2+yz-xy=2$......(5).

Factoring $(y-x)(y+x+z)=1$......(6).

$(z-x)(z+x+y)=2$......(7). Whence $y=\dfrac{x+z}{2}$.

Substituting in (1) and (3) and letting $z=vx$,

$$x^2=\frac{2}{2-v-v^2}=\frac{6}{2v^2-v-1}\text{......(8).}$$

Whence $v=1$ or $-\frac{7}{5}$..... (9).

For $v=1$, leads to indeterminate results.

For $v=-\frac{7}{5}$, $x=\pm\frac{5}{6}\sqrt{2}$, $y=\pm\frac{1}{6}\sqrt{2}$, $z=\pm\frac{7}{6}\sqrt{2}$.

V. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $y=ax$, $z=bx$.

$\therefore$ $x^2-abx^2=1$, $a^2x^2-bx^2=2$, $b^2x^2-ax^2=3$.

$\therefore$ $x^2=1/(1-ab)=2/(a^2-b)=3(b^2-a)$.

$\therefore$ $a^2-b=2-2ab$, $b^2-a=3-3ab$.

$\therefore$ $b=(a^2-2)/(1-2a)$.

$\therefore$ $5a^4+a^3-5a-1=0$.

$\therefore$ $(5a^4+1)(a^3-1)=0$.

$\therefore$ $a=-\frac{1}{5}$, $a=1$, $a=-\frac{1}{2}(1\mp\sqrt{-3})$, $b=-\frac{7}{5}$, $b=1$, $b=-\frac{1}{2}(1\pm\sqrt{-3})$.

The second and third values of a and b give $x=y=z=\infty$ and are therefore not admissible.

$a=-\frac{1}{5}$, $b=-\frac{7}{5}$, give $x=\pm\dfrac{5}{3\sqrt{2}}$, $y=\mp\dfrac{1}{3\sqrt{2}}$, $z=\mp\dfrac{7}{3\sqrt{2}}$.

Also solved by *P. S. BERG, J. M. BOORMAN, W. H. DRANE, ALOIS F. KOVARIK, CHARLES E. MEYERS, H. N. HERRICK, NELSON L. RORAY, E. D. SCALES*, and the *PROPOSER*.

GEOMETRY.

112. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

The tangent planes at A, B, C, D to the sphere circumscribing the tetrahedron $ABCD$ form a tetrahedron $abcd$; prove that Aa, Bb, Cc, Dd will meet in a point if $BC.AD=CA.BD=AB.CD$.

I. Solution by the PROPOSER.

Using tetrahedral coördinates, the equation to the sphere circumscribing the tetrahedron of reference is

$$2a^2\beta\gamma+2b^2\alpha\gamma+2c^2\alpha\beta+2a'^2\alpha\delta+2b'^2\beta\delta+2c'^2\gamma\delta=0\ldots\ldots(1),$$

a, b, c being the sides of the base of the tetrahedron, and a', b', c' the edges opposite.

The polar plane of A (1, 0, 0, 0) is $c^2\beta+b^2\gamma+a'^2\delta=0\ldots\ldots(2)$,

of B, $c^2\alpha+a^2\gamma+b'^2\delta=0\ldots\ldots(3)$,

of C, $b^2\alpha+a^2\beta+c'^2\delta=0\ \ldots.(4)$,

and of D, $a'^2\alpha+b'^2\beta+c'^2\gamma=0\ldots\ldots(5)$.

Now *assuming*, for simplicity, $aa'=bb'=cc'\ldots\ldots(6)$, and calling the vertices of the tetrahedron opposite (2), (3), (4), (5), a, b, c, d, we have for the coördinates of a,

$$\alpha_1=\frac{2a^4a'^2}{a^2b^2c^2-2a^4c'^2+a^2c^2a'^2+a^2b^2a'^2}=-\frac{2a^4a'^2}{D_1}\ldots\ldots(7).$$

$$\beta_1=\frac{b^4b'^2}{D_1}\ldots\ldots(8),\quad \gamma_1=\frac{c^4c'^2}{D_1}\ldots\ldots(9),\quad \delta_1=\frac{a^2b^2c^2}{D_1}\ldots\ldots(10).$$

Then the equation to the line Aa is

$$\frac{1-\alpha}{a^2b^2c^2+b^2d^2a'^2+c^2a^2a'^2}=\frac{\beta}{a^2b^2a'^2}=\frac{\gamma}{a^2c^2a'^2}=\frac{\delta}{a^2b^2c^2}\ldots\ldots(11).$$

Similarly, the equations to Bb, Cc, Dd, are, respectively,

$$\frac{\alpha}{a^2b^2b'^2}=\frac{1-\beta}{a^2b^2c^2+a^2b^2b'^2+b^2c^2b'^2}=\frac{\gamma}{b^2c^2b'^2}=\frac{\delta}{a^2b^2c^2}\ldots\ldots(12).$$

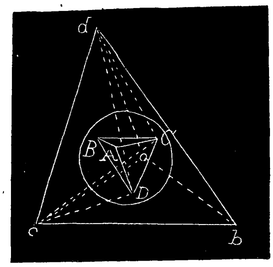
$$\frac{\alpha}{a^2c^2c'^2}=\frac{\beta}{b^2c^2c'^2}=\frac{1-\gamma}{a^2b^2c^2+b^2c^2c'^2+a^2c^2c'^2}=\frac{\delta}{a^2b^2c^2}\ldots\ldots(13),$$

$$\frac{\alpha}{a^2b^2b'^2}=\frac{\beta}{b^2c^2c'^2}=\frac{\gamma}{a^2c^2a'^2}=\frac{1-\gamma}{a^2b^2b'^2+b^2c^2c'^2+a^2c^2a'^2}\ldots\ldots(14).$$

(11), (12), (13), (14) are concurrent.

II. Solution by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

Let $ABCD$ be the tetrahedron about which is circumscribed the sphere ; and $abcd$ the tetrahedron formed by planes tangent respectively at A, B, C, and D. Given, also, that $BC.AD=CA.BD=AB.CD$.



To prove that Aa, Bb, Cc, Dd meet in a point, as at O.

The points c, D, C, and d are in the same plane. For D and C have their respective distances from the planes cbd and cad in the same ratio, viz : AD^2/AC^2 and BD^2/BC^2, which are equal, since by hypothesis $BC.AD=AC.BD$.

That first part of preceding statement is true is evident from the fact that if diameters are supposed to be drawn from A and B respectively, the respective projections upon these diameters of the chords AD, AC, and BD, BC, have the same ratios respectively as the squares of the chords ; and the projections equal the distances respectively from the points D and C to the planes.

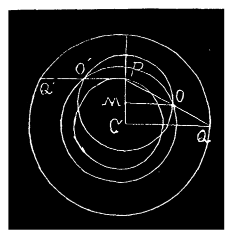
Similarly, co-plainar are the points c, B, C, b ; b, A, B, a ; and a, D, A, d.

Therefore, Dd and Cc intersect ; so, also, Dd and Bb, and Cc and Bb.

113. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama.

Given three concentric circles. Draw a straight line from the inner to the outer circumference that shall be bisected by the middle circumference.

I. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O., and J. C. GREGG, A. M., Superintendent of Schools, Brazil, Ind.



Let C be the common center and CP the radius of the inner circle, and CQ that of the outer circle.

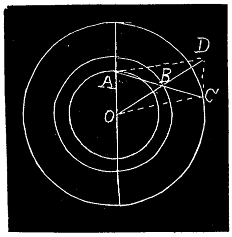
Bisect CP at M and with one-half of CQ as radius and M as center describe a circle cutting the middle circumference at O or O'. Draw PO and produce to the outer circumference at Q. Then POQ is the required line. For, $\because PM=MC$ and $MO=\frac{1}{2}CQ$, $\therefore PO=OQ$.

According as $\frac{1}{2}$radius of outer circle is greater than, equal to, or less than the radius of the middle circle increased by $\frac{1}{2}$radius of inner circle, there are two solutions, one solution, or no solution.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; J. SCHEFFER, A. M., Hagerstown, Md.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; J. C. NAGLE, C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Tex.; CHAS. C. CROSS, Libertytown, Md.; J. O. MAHONEY, B. E., M. Sc., Teacher of Mathematics and Science in Carthage High School, Carthage, Tex., and the PROPOSER.

Let O be the center of the circles, and A any point on the inner circumference. With O as a center and a radius$=$2 times the radius of the middle circle, and with A as a center and a radius$=$the radius of the outer circle, describe two arcs intersecting at D. Draw OD intersecting middle circumference at B.

Through A and B draw AC intersecting outer circumference at C. Then $AB=BC$, and AC is the required line.



PROOF. $OB=BD$, $OC=AD$, and $\angle OBC=\angle ABD$. $\therefore \triangle OBC=\triangle ABD$. $\therefore AB=BC$. Also, $OADC$ is a parallelogram, of which the diagonals OD and AC bisect each other at B.

COROLLARY 1. Put a, b, and c=the respective radii of the three concentric circles, taking $a<b<c$, and put $2d$=line AC. Then, from the relation of the diagonals to the sides of a parallelogram,

$$2d=\sqrt{(2c^2+2a^2-4b^2)}.$$ GRUBER.

COROLLARY 2. The problem is possible only for $c-b=b-a$ to $c-b=b+a$, or for $2b=c+a$ to $2b=c-a$. Whence the limits of $2d$ are $c-a$ and $c+a$, the parallelogram in either case reducing to a straight line. GRUBER.

COROLLARY 3. The point D is without the outer circle for $2b>c$. When $2b=c$, D lies in the outer circumference. When $2b<c$, D lies within the outer circle. GRUBER.

COROLLARY 4. When $2b>\sqrt{(c^2+3a^2)}$, $2d$ lies wholly without the inner circle. When $2b=\sqrt{(c^2+3a^2}$, $2d$ is tangent to the inner circumference. When $2b<\sqrt{(c^2+3a^2)}$, $2d$ is a secant of the inner circle. GRUBER.

Also solved by *G. B. M. ZERR*, *J. W. YOUNG*. *GEORGE R. DEAN*, *B. F. YANNEY*, *B. F. SINE*, *WALTER H. DRANE*, and *WM. K. NORTON*.

CALCULUS.

88. Proposed by **JOHN M. ARNOLD**, Crompton, R. I.

When a watch is wound up, the mainspring is closely coiled around a cylindrical piece called the hub of the barrel-arbor. When entirely run down the spring forms an annulus against the inner circumference of the barrel. Show that if the width of the annulus is a little more than one-fourth of the radius of the barrel, the spring will run the watch the greatest number of hours at one winding, the diameter of the hub being one-third the inside diameter of the barrel.

I. Solution by the **PROPOSER**.

Let R=radius of the barrel, r=radius of the hub, t=thickness of spring, x=width of the annulus when run down, y=width of the annulus when wound up, u=number of turns required to wind the spring.

Then x/t=number of coils of the spring when run down, and y/t=number of coils when wound up.

Hence $u=y/t-x/t$......(1).

It is evident that the area on the bottom of the barrel covered by the spring will be the same in the wound or unwound condition.

Hence $R^2\pi-(R-x)^2\pi-(r+y)^2\pi-r^2\pi$.

Reducing $2Rx-x^2=2ry+y^2$..... (2).

From (1) and (2), $tu=\pm\sqrt{[2Rx-x^2+r^2]}-r-x$.

Differentiating and equating to zero,

$$\frac{du}{dx}=\pm\frac{R-x}{t\sqrt{[2Rx-x^2+r^2]}}-1/t=0.$$

Reducing, $2x^2-4Rx=r^2-R^2$.

Whence $x=R\pm\sqrt{\left(\frac{R^2+r^2}{2}\right)}$.

Making $r=\frac{1}{3}R$, and taking the minus sign, $x=R(1-\sqrt{\frac{5}{9}})$.

$x=.25464R$, or a little more than one-fourth of the radius.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Ia.

To secure the greatest result, the area occupied by the spring when wound or unwound must be one-half that between the hub and the inner circumference of the barrel. This area is $\frac{8}{9}r^2\pi$, and the area occupied by the hub and spring when the latter is wound, is $\frac{5}{9}r^2\pi$. Hence the radius of the circumference lying within the annulus is $r'=\frac{1}{3}r\sqrt{5}=.745r$.

$\therefore r-r'=.254r$.

III. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

Let a=area of the cross-section of the barrel; then $(\frac{1}{3})^2a=\frac{1}{9}a$=the area of the cross-section of the hub; $\frac{8}{9}a$=area around the hub, and $\frac{4}{9}a$=one-half of that area; $\frac{5}{9}a$=area of cross-section of hub and spring.

Hence both hub and spring occupy $\sqrt{\frac{5}{9}}=.7454$ of the radius of the barrel, and the unwound spring occupies $1-.7454=.2546$ of that radius.

MECHANICS.

81. Proposed by JAMES S. STEVENS, Professor of Physics, The University of Maine, Orono, Me.

Two iron spheres whose weights are a and b, and a is greater than b, are suspended over a frictionless pulley so that they move in a liquid medium of density δ. Assume that the density of the iron is δ', what would be the spaces passed over (downward by a and upward by b) in the first four seconds, if the spheres start from rest?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let m=the mass$=(a-b)/g$......(1).

v=the velocity at time t, Rv^2=resistance. The resistance is the sum of the resistances for both spheres.

Let A=the greatest cross-section of $a=\dfrac{\sqrt[3]{[36a^2\pi g\delta'}}{4g\delta'}$.

Let B=the greatest cross-section of $b=\dfrac{\sqrt[3]{[36b^2\pi g\delta'}}{4g\delta'}$.

From Rankine, $Rv^2=\dfrac{kev^2}{2g}(A+B)$ where $k=0.51$ for the sphere, $e=$the weight of unit of fluid$=g\delta$.

$$\therefore R=\frac{0.51\delta\sqrt[3]{[36\pi g\delta'][\sqrt[3]{(a^2)}+\sqrt[3]{(b^2)}}}{8g\delta'}\ldots\ldots(2).$$

Equation of motion is, $m(dv/dt)=mg-Rv^2$.

$$\therefore t=\frac{m}{2\sqrt{(gmR)}}\log_e\left(\frac{\sqrt{(mg)}+v\sqrt{R}}{\sqrt{(mg)}-v\sqrt{R}}\right).\quad \text{Let } n=\sqrt{(gR/m)}\ldots\ldots(3).$$

$$\therefore v=(dx/dt)=\frac{mn}{R}\left(\frac{e^{nt}-e^{-nt}}{e^{nt}+e^{-nt}}\right).\quad \therefore x=\frac{mn}{R}\int_0^4\left(\frac{e^{nt}-e^{-nt}}{e^{nt}+e^{-nt}}\right)dt.$$

$$\therefore x=\frac{m}{R}\log_e\left(\frac{e^{4n}+e^{-4n}}{2}\right)\ldots\ldots(4).$$

(1), (2), (3) in (4) gives the result required.

Let $g=32.16$, $\delta=1$, $\delta'=7.8$, $a=40$ lbs., $b=7.84$ lbs.

$\therefore m=1$, $R=.1212$, $n=1.974$.

$\therefore x=8\frac{1}{3}\log_e\left(\dfrac{e^{7.896}+e^{-7.896}}{2}\right)=8\frac{1}{3}\log_e(1343.3184)$.

$\therefore x=60.0233$ feet.

II. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Let T be the tension of the string.

The mass of the liquid displaced by the heavier body is ad/gd', of that displaced by the lighter body is bd/gd'; measuring distances downward for the heavier body and upward for the lighter body, assuming that resistance of the liquid to motion varies as the square of the velocity, the equations of motion for the heavier and the lighter body are respectively:

$$a\frac{d^2s}{dt^2}=ag-\frac{ad}{d'}-kv^2-gT\ldots\ldots(1).$$

$$b\frac{d^2s}{dt^2}=-bg+\frac{bd}{d'}-kv^2+gT\ldots\ldots(2).$$

$$\text{Eliminating } T,\ \frac{d^2s}{dt^2}=\frac{(a-b)(d'g-d)}{(a+b)d'}-2kv^2\ldots\ldots(3).$$

By integration twice :

$$s=\frac{(a+b)d'}{(a-b)(d'g-d)}\log\left[\frac{l\ t^{\sqrt{[2k(a-b)(d'g-d)]}/\sqrt{[(a+b)d']}}+l^{-t\sqrt{[2k(a-b)(d'g-d)]}/\sqrt{[(a+b)d']}}}{2}\right]$$

$$\text{or } s=\frac{(a+b)d'}{(a-b)(d'g-d)}\log(\cosh t\sqrt{[2k(a-b)(d'g-d)]}/\sqrt{[(a+b)d']}).$$

$$s=\frac{1}{p^2}\log(\cosh pt) \text{ where } p=\sqrt{\frac{2k(a-b)(d'g-d)}{(a+b)d'}}.$$

For four seconds, $s=\frac{1}{p^2}\log(\cosh 4p)$.

[See Bowser's *Analytic Mechanics*, page 314, ex. 5, where $v=0$ and $d=0$, of equation (3) above.]

Also solved by *ELMER SCHUYLER*.

AVERAGE AND PROBABILITY.

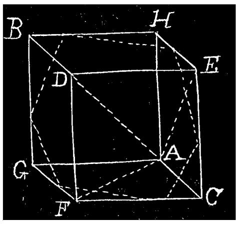
61. Proposed by COL. CLARKE.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? [Erom Williamson's *Integeal Calculus*.]

Solution by LEWIS NEIKIRK, Graduate Student, University of Colorado, Boulder, Col.

I. PRELIMINARY INVESTIGATION.

Let S, the random section, be determined by the coördinates p, φ, and θ, θ being the angle between p and its projection on $ACFG$ and φ the angle between AC and the projection of p. Let P be the point of intersection of p and S. Also let p increase from zero for φ and $\theta<\frac{1}{2}\pi$. S, starting with three sides at A, gains three more, one at each of the corners C, G, and H; and loses three, one at each of the corners B, E, and F. φ and θ determine the order in which these gains and losses shall occur, and plainly S can be hexagonal only when the first loss is antedated by all three gains.



For p in the diagonal AD ($\varphi=\frac{1}{4}\pi$, $\theta=\cot^{-1}\sqrt{2}$), the three gains are simultaneous and are followed by three simultaneous losses. For p as an element of the area DAF ($\varphi=\frac{1}{4}\pi$, $\theta<\cot^{-1}\sqrt{2}$), one gain at H is followed by two more at C and G; then two losses at B and E, followed by one loss at F. For p as an element of the areas DAE and DAB ($\varphi<\frac{1}{4}\pi$ and $\theta=\tan^{-1}\cos\varphi$, and $\varphi>\frac{1}{4}\pi$ and $\theta=\tan^{-1}\sin\varphi$) there is a like sequence of gains and losses. So far it has been easy to enumerate the exact order in which all the gains and losses occur.

For p within the solid angle $A—DECF$ ($\varphi<\frac{1}{4}\pi$ and $\theta<\tan^{-1}\cos\varphi$) the first loss occurs at B, and the last gain at C; the order and place of the remaining gains and losses would be difficult to enumerate and is in any event immaterial

to the solution. A similar enumeration and statement to the angles $A—DBGF$ and $A—DBHE$. Evidently, then, our attention may be confined to any one of these angles, say $A—DECF$.

II. SOLUTION.

As p increases within the solid angle $A—DECF$ ($\varphi<\frac{1}{4}\pi$ and $\theta<\tan^{-1}\cos\varphi$), S, starting at A with three sides, gains two sides successively at G and H, the order being immaterial; then for some values of φ and θ it will gain a third side at C before losing the first one at B; for other values this loss at B will andedate the third gain at C; in fact it may antedate the second of the two gains at G or H. For certain values of φ and θ between these extremes, this first loss and final gain will concur. In this last case p is an element of the area EAF ($\varphi<\frac{1}{4}\pi$ and $\theta=\tan^{-1}[\cos\varphi-\sin\varphi]$); for this area is plainly perpendicular to BC, and must therefore contain p when S reaches B and C simultaneously.

The number of cases which have the coördinates p, φ, and θ are $dpd\omega=\cos\theta dpd\theta d\varphi$, where ω is a solid angle with its vertex at A. The integration of p for the favorable cases extends from p_1 to p_2, where p_1 is the value of p when S reaches C, and p_2 is the value of p when S reaches B. The integration of θ extends from the plane EAF to the plane DAE, and the integration of φ from 0 to $\frac{1}{4}\pi$. The above limits may be calculated from the following spherical triangles in which the primed letters refer to points on a unit sphere, center at A, corresponding to points with unprimed letters in the figure.

It the right spherical triangle $P'C'F'$, $P'C'=\psi_1$, $F'C'=\varphi$, $P'F'=\theta$, and $\angle P'F'C'=90°$. Then $p_1=a\cos\psi_1=a\cos\theta\cos\varphi$, where a is an edge of the cube.

In the spherical triangle $B'P'H'$. $B'F'=45°$, $B'P'=\psi_2$, $P'F'=90°-\theta$, and $\angle B'H'P'=90°-\varphi$. Then $p_2=a\sqrt{2}\cos\psi_2$, $\cos\psi_2=(1/\sqrt{2})(\sin\theta+\cos\theta\sin\varphi)$.

$\therefore\ p_2=a(\sin\theta+\cos\theta\sin\varphi)$.

In the right spherical triangle $F'P'C'$ right angled at C', $P'C'=\theta_1$, $F'C'=45°-\varphi$. and $\angle P'F'C'=\cot^{-1}(1/\sqrt{2})$.

Then $\tan\theta_1=\sqrt{2}\sin(45°-\varphi)=\cos\varphi-\sin\varphi$. Then $\theta_1=\tan^{-1}(\cos\varphi-\sin\varphi)$.

In the right spherical triangle $P'H'E'$ right angled at E', $H'E'=45°$, $P'H'=90°-\theta_2$, and $\angle P'H'E'=\varphi$. Then $\tan\theta_2=\cos\varphi$, $\theta_2=\tan^{-1}\cos\varphi$.

All the favorable cases

$$F=12\int_0^{\frac{1}{4}\pi}\int_{\theta_1}^{\theta_2}\int_{p_1}^{p_2}\cos\theta dpd\theta d\varphi=12a(\sqrt{3}\tan^{-1}\tfrac{1}{3}\sqrt{3}-\sqrt{2}\tan^{-1}\tfrac{1}{2}\sqrt{2}).$$

The integration for the total number of cases extends for p from 0 to p', where p' is the value of p when S reaches D; for θ, from 0 to $\frac{1}{2}\pi$, and for φ from 0 to $\frac{1}{2}\pi$.

In the spherical triangles $D'AP'$ and $D'AF'$, $D'A=\psi'$, $P'A=\theta$, $D'P'=\beta$, $\angle D'AP'=\alpha$, $F'A=(45°-\varphi)$, $F'D'=\tan^{-1}(1/\sqrt{2})$, $\angle D'F'A=90$, and $\angle D'AF'=(90°-\alpha)$.

Then $\cos\psi'=\sqrt{\tfrac{2}{3}}\cos(45°-\varphi)=(1/\sqrt{3})(\cos\varphi+\sin\varphi)$, $\tan\alpha=\sqrt{2}\sin(45-\varphi)$
$=\cos\varphi-\sin\varphi$,

$p = a\sqrt{3}\,\cos\beta = a\sqrt{3}\,(\cos\psi'\cos\theta + \sin\psi'\sin\theta\cos\alpha)$
$= a[\cos\theta(\cos\varphi + \sin\varphi) + \sin\theta]$.

The total number of cases is

$$T = 4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{p'}\cos\theta\, dp\, d\theta\, d\varphi = 3\pi a.$$

Therefore the probability is $P = E/T = 4/\pi(\sqrt{3}\,\tan^{-1}\tfrac{1}{3}\sqrt{3} - \sqrt{2}\,\tan^{-1}\tfrac{1}{2}\sqrt{2})$.

NOTE.—I wish to acknowledge my indebtedness to Professor DeLong and Mr. Frank Giffin for looking over this solution and making valuable suggestions.

MISCELLANEOUS.

70. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Prove $\tan^{-1}x = \dfrac{1}{2i}\left(\log\dfrac{x-i}{x+i}\right)$, and thence that $\pi = (2/i)\log(i)$.

I. Solution by GUY B. COLLIER and HAROLD C. FISKE, Class 1901, Union College, Schenectady, N. Y., and the PROPOSER.

Consider the integral,

$$\int\frac{dx}{1+x^2} = \tan^{-1}x\ldots\ldots(1).$$

Integrate the left member by partial fractions

$$\int\frac{dx}{1+x^2} = \frac{1}{2i}\int\frac{dx}{x-i} - \frac{1}{2i}\int\frac{dx}{x+i} = \frac{1}{2i}\log\frac{x-i}{x+i}\ldots\ldots(2).$$

$\therefore$ From (1) and (2),

$$\tan^{-1}x = \frac{1}{2i}\log\frac{x-i}{x+i}.$$

When $x=1$ this becomes

$$\tfrac{1}{4}\pi = \frac{1}{2i}\log\frac{1-i}{1+i} = \frac{1}{2i}\log(-i).$$

$\pi = (2/i)\log(-i)$, it should have been.

II. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

We have the identity

$$\frac{1}{1+x^2} = \frac{1}{2i}\left(-\frac{1}{i-x} - \frac{1}{i+x}\right).\ \therefore\int\frac{dx}{1+x^2} = \frac{1}{2i}\log\left(\frac{i-x}{i+x}\right) + c.$$

$$\text{But} \int \frac{dx}{1+x^2} = \tan^{-1}x + c. \quad \therefore \tan^{-1}x = \frac{1}{2i} \log\left(\frac{i-x}{i+x}\right).$$

Making $x=0$, it is seen that no constant need be added.

Now making $x=1$, we have

$$\tfrac{1}{4}\pi = \frac{1}{2i}\log\left(\frac{i-1}{i+1}\right) = \frac{1}{2i}\log\left(\frac{-2i}{-2}\right) = \frac{1}{2i}\log(i). \quad \therefore \pi = \frac{2}{i}\log(i).$$

Or we may obtain this from the equation $e^{i\theta} = \cos\theta + i\sin\theta$ by putting $\theta = \frac{1}{2}\pi$ and taking logarithm of both sides.

(The proposer of this problem seems to have neglected the matter of determining the constant and so has the sign under the logarithm wrong.)

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$$\frac{1}{1+x^2} = 1/2i.\frac{(x+i)-(x-i)}{1+x^2} = 1/2i\frac{[(x+i)-(x-i)](x+i)}{(x+i)^2(x-i)}. \quad \text{When } i = \sqrt{(-1)}.$$

$$\therefore \frac{dx}{1+x^2} = 1/2i.\frac{[(x+i)-(x-i)]\,(x+i)dx}{(x+i)^2(x-i)}. \quad \therefore \tan^{-1}x = 1/2i.\left(\log\frac{x-i}{x+i}\right).$$

Let $x=1$. $\therefore \frac{1}{4}\pi = (1/2i)\log\dfrac{1-i}{1+i} = (1/2i)\log(-1)$. $\therefore \pi = (2/i)\log(-i)$.

IV. Solution by J. O. MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Texas.

Let $y+ix=re^{ia}$ ..... (1), and $y-ix=re^{-ia}$ ......(2).

Then $r^2=x^2+y^2$, $\tan a = x/y$.

From (1) and (2), $\log(y+ix)-\log(y-ix)=2ia=2i\tan^{-1}(x/y)$, or

$$\log\frac{y+ix}{y-ix} = 2i\tan^{-1}(x/y); \text{ whence } \log\left(\frac{(y/i)+x}{(-y/i)+x}\right) = 2i\tan^{-1}(x/y),$$

$$\text{or } \log\left(\frac{x-iy}{x+iy}\right) = 2i\tan^{-1}(x/y).$$

Put $y=1$, then $(1/2i)\log\dfrac{x-i}{x+i} = \tan^{-1}x$.

If $x=0$, then $\tan^{-1}0 = \pi = (1/2i)\log(-i/i)$, or $\pi = (1/2i)\log(i)$.

V. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Putting in $\log(a+bi)$, $a=\rho\cos\phi$, $b=\rho\sin\phi$ ... .(1).

Whence $\rho=(a^2+b^2)^{\frac{1}{2}}$ ..... (2), $\phi=\tan^{-1}(b/a)$ ....(3).

We have $\log(a+bi)=\log\rho+\log(\cos\phi+\sin\phi\sqrt{-1})=\log\rho+\log e^{\phi\sqrt{-1}}$

$$=\log\rho+\phi\sqrt{-1}=\tfrac{1}{2}\log(a^2+b^2)+i\tan^{-1}(b/a).$$

$\therefore\ \tan^{-1}(b/a)=(1/i)[\log(a+bi)-\tfrac{1}{2}\log(a^2+b^2)]$

$$=(1/i)\log\frac{a+bi}{(a^2+b^2)^{\frac{1}{2}}}=(1/2i)\log\frac{(a+bi)^2}{a^2+b^2}=(1/2i)\log\frac{a+bi}{a-bi}$$

$$\text{sin}a^2+b^2=(a+bi)(a-bi).$$

Putting now $b=x$, $a=1$, we have

$$\tan^{-1}x=(1/2i)\log\frac{1+ix}{1-ix},$$

or multiplying numerator and denominator of the fraction under the logarithmic function by i, we have

$$\tan^{-1}x=(1/2i)\log\frac{i-x}{i+x}.$$

Putting $x=1$, we have $\tfrac{1}{4}\pi=(1/2i)\log\dfrac{i-1}{i+1}=(1/2i)\log\dfrac{(i-1)^2}{(i+1)(i-1)}$

$$=(1/2i)\log(-2i/-2)=(1/2i)\log i.\quad \therefore\ \pi=(2/i)\log i.$$

NOTE.—There is a slight error in the statement, since $x-i$ should be $i-x$.

VI. Solution by COOPER D. SCHMITT, A.M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn., and ELMER SCHUYLER, High Bridge, N. J.

From trigonometry we have the formula

$$\log[\alpha+\beta i)=\log\sqrt{(\alpha^2+\beta^2)}+i\tan^{-1}(\beta/\alpha).$$

Let $\alpha=x$, $\beta=-1$, and we have

$$\log(x-i)=\log\sqrt{(x^2+1)}+i\tan^{-1}(-1/x).$$

Let $\alpha=x$, $\beta=1$, and we have

$$\log(x+i)=\log\sqrt{(x^2+1)}+i\tan^{-1}(1/x).$$

Subtracting, we have

$$\log\frac{x-i}{x+i}=i[\tan^{-1}(-1/x)-\tan^{-1}(1/x)=i\tan^{-1}\left(\frac{-(1/x)-(1/x)}{1-(1/x^2)}\right)$$

$$=i\tan^{-1}\frac{2x}{1-x^2}=2i\tan^{-1}x.$$

Now let $x=1$, and we have

$$2i\tfrac{1}{4}\pi=\log\frac{1-i}{1+i}=\log i,\ \text{or}\ i\pi=2\log i,\ \pi=(2/i)\log i.$$

Or further, $i\pi=\log i^2=\log(-1)$, or $\pi=\dfrac{\log(-1)}{\sqrt{(-1)}}$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

116. Proposed by J. O. MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.

Two candles are of the same length. The one is consumed uniformly in 4 hours, and the other in 5 hours. If the candles are lighted at the same time, when will one be three times as long as the other?

117. Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey, 1905 Sixteenth St., Washington, D. C.

A landed man two daughters had,
And both were very fair;
He gave to each a piece of land,
One round the other square.

At twenty shillings an acre, just,
Each piece its value had;
The shillings that did compass each,
For it exactly paid.

If 'cross a shilling be an inch,
(As it is, very near),
Which had the larger portion, she
That had the round or square?

Also, how many acres did each receive?
[Does any one know the history of this problem?]

*** Solutions of these problems should be sent to B. F. Finkel not later than Sept. 10.

ALGEBRA.

105. Proposed by CHARLES E. MYERS, Canton, Ohio.

Solve for x the following: $a\log(ax^2)=m\log(m)$.

106. Proposed by ELMER SCHUYLER, High Bridge, N. J.

$$\frac{x^2+x}{y^2+y}=a;\quad \frac{x^2+y}{y^2+x}=b;\text{ find } x \text{ and } y.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than Sept. 10.

GEOMETRY.

124. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics, Drury College, Springfield, Mo.

Every conic that passes through all the foci of a conic is a rectangular hyperbola. [From Charlotte A Scott's *Modern Analytical Geometry*.]

125. Proposed by J. SHEFFER, A. M., Hagerstown, Md.

To find the locus of a point on the surface of an ellipsoid which has the property that the tangent plane at that point is at the given distance, f, from the center of the ellipsoid.

*** Solutions of these problems should be sent to B. F. Finkel not later than Sept. 10.

CALCULUS.

95. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A ship starts at the equator and sails northeast at all times. How far has the ship sailed (in miles) when her latitude is 30°, 45°, 60°, 90° ? How far when her longitude is 90°, 180°, 270°, 360° ? Regarding the earth as a sphere, radius 3956 miles.

96. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

If $f(x)=\int f(x)dx$, find $f(x)$, the constant being zero.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Sept. 10.

MECHANICS.

93. Proposed by WALTER H. DRANE, Graduate Student, Harvard University.

A small rope, which is passed over a smooth pulley, has attached at one end a weight of twenty pounds and at the other end hangs a monkey, also weighing twenty pounds. Is it possible for the monkey to climb to the pulley, and if so, what will happen to the weight?

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In a parallelogram $ABCD$, $\angle D=\beta$, $AB=a$, $BC=b$, the principal moments of inertia at the centroid are $(\frac{1}{24}m)[a^2+b^2\pm\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}]$ and the principal axes at the same point make with the side CD an angle θ given by

$$\tan2\theta=\frac{b^2\sin2\beta}{a^2+b^2\cos2\beta}.$$

95. Proposed by FLORIAN CAJORI, Ph. D., Author of History of Mathematics, History of Physics, etc., and Professor of Mathematics, Colorado College, Colorado Springs, Colorado.

Assuming that the velocity is proportional to the distance described from the state of rest, (1) can the body start in motion? (2) If it can, what is its initial acceleration? If we make the additional assumption that the time of fall, from rest, through a finite distance is finite, does it follow that the velocity is infinite?

96. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Two particles, subject to their mutual attraction and that of a fixed center, move in a plane containing the center. Find the motion under the law of the inverse square.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Sept. 10.

DIOPHANTINE ANALYSIS.

81. Proposed by A. H. BELL, Hillsboro, Ill.

Given $2x^2-47y^2=-29$. To find four integral values for x and y.

82. Proposed by J. H. DRUMMOND, LL. D., Portland, Me.

In the series $1^3+3^3+5^3$ ...... find n so that the nth term and the sum of n terms shall both be squares.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Sept. 10.

AVERAGE AND PROBABILITY.

77. Proposed by J. O. MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.; and ELMER SCHUYLER, Annapolis, Md.

A and B are two inaccurate arithmeticians whose chance of solving a given question correctly is 1/8 and 1/12 respectively; if they obtain the same result, and if it is 1000 to 1 against their making the same mistake, find the chance that the result is correct. [*From Hall and Knight's Algebra.*]

78. Proposed by CHAS. E. MYERS, Canton, O.

Two witnesses, A and B, both make the statement that an event happened in a particular way (two ways being possible). Find the probability of the truth of the statement.

79. Proposed by the late ENOCH BEERY SEITZ.

Two equal spheres touch each other externally. If a point be taken at random within each sphere, show that (1) the chance that the distance between the points is less than the diameter of either sphere is 13/35, and (2) the average distance between them is $11/5r$. [This is Problem 5835, *Educational Times*, of London.]

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Sept. 10.

BOOKS AND PERIODICALS.

Bibliography of American Arithmetics. By J. M. Greenwood, Superintendent of Schools, Kansas City, Mo., and Artemas Martin, A. M., Ph.D., LL. D., Geodetic Survey Office, Washington, D. C. Issued by the National Bureau of Education.

This is a very complete bibliography of the Arithmetics published in this country from the earliest time down to the present. Not only the names of the book, the author and the publishers are given, but also a brief description of each book. B. F. F.

A Primer of the Calculus. By E. Sherman Gould, Member American Society of Civil Engineers. Second edition, revised and enlarged. 16mo. Cloth, 122 pages. Price, 50 cents. New York: D. Van Nostrand Company.

This work is a development of the infinitesimal method of the calculus. It is restricted in its treatment to the absolute rudiments of the science, being intended for practical use rather than for class-room work. The favorable reception of the first edition has made it necessary to bring out a second edition which the author has somewhat enlarged and improved. B. F. F.

The Ranger. An Instrument Invented and Patented by Francis J. Bayldon, Extra Master, and A. Huddart Armstrong, Engineer, for the use of Seamen while Navigating their Vessels off the Land. S. E. Lees, Printer and Stationer, 81 Clarence Street, Sydney, Australia.

In this pamphlet of 16 pages, is described 'The Ranger," an instrument for measuring triangles, that at once shows the value of the sides and angles of any triangle laid off on it, without the least calculation being necessary. The instrument is specially adapted to the use of seamen when in sight of land. It is very simple in construction and accurate in its results. B. F. F.

School Algebra with Exercises. By George Egbert Fisher, M. A., Ph. D., and Isaac J. Schwatt, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania. 8vo. Cloth, 406 pages. Price, \$1.00. Philadelphia: Fisher and Schwatt.

This book retains the distinctive features of the authors' Text-Book of Algebra, Part I., but written with a view to the needs of younger students. "The aim has been to make the transition from ordinary Arithmetic to Algebra natural and easy. Nothing has been slighted or evaded, and all difficulties have been honestly faced and explained. Special attention has been paid to making clear the reason for every step taken. Each principle is first illustrated by particular examples, thus preparing the mind of the student to grasp the meaning of a formal statement of the principle and its proof. . . . The importance of mental discipline to every student of Mathematics has also been fully recognized. On this account great care has been taken to develop the subject in a logical manner. Rigorous, but, as a rule, simple proofs of all principles have been given." The book is in every detail one of the highest merit and is worthy the patronage of all teachers of Algebra.

B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, \$2.50 per year in advance. Single numbers, 25 cents.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, \$1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

ERRATA.

In the last paragraph of "Note on Right Triangles," THE AMERICAN MATHEMATICAL MONTHLY, Vol. VI, No. 3, pp. 91, 92, we find, "A given area, or a given perimeter, can belong to but one prime right-angled triangle."

How will Professor Shedd reconcile the two prime right triangles whose respective sides are 12, 35, 37, and 20, 21, 29? Area$=210=\frac{1}{2}\times12\times35=\frac{1}{2}\times20\times21$.

M. A. GRUBER.

On pages 138 and 139 of THE AMERICAN MATHEMATICAL MONTHLY (May), there is a misprint which may be corrected easily by the reader. It is obvious that the tangent of the angle included by the two lines is

$$\frac{\dfrac{n}{m}\cdot\dfrac{r}{\sqrt{R^2-r^2}}+\dfrac{n_1}{m_1}\cdot\dfrac{r}{\sqrt{R^2-r^2}}}{1-\dfrac{n}{m}\cdot\dfrac{n_1}{m_1}=\dfrac{r^2}{R^2-r^2}},$$

so that the condition for a right angle (formula 10, pp. 138 and 139), is

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{R^2-r^2}{r^2}.$$

A. EMCH.

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BIOGRAPHY.

DR. PERCIVAL FROST.

BY DR. GEORGE BRUCE HALSTED.

PERCIVAL FROST made one in that Cambridge paradox, Second Wranglers greater than their Seniors.

In 1837 the great Sylvester was second to Wm. N. Griffin, both of St. John's College ; in 1839 Frost was second to B. M. Cowie, where the first four places in the Mathematical Tripos were all won by men of St. John's College. Of other great seconds we recall Dr. Whewell, Clerk Maxwell, Lord Kelvin, Clifford, J. J. Thomson, truly a goodly company !

But Frost was First Smith's Prizeman, a contest at which nearly all these great seconds beat their firsts, though Sylvester, being unwilling to sign the Thirty-nine Articles, was debarred from entering this cnmpetition.

Percival Frost, second son of Charles Frost, F. S. A., a solicitor practicing in the town of Kingston-upon-Hull, was there born on September 1, 1817. He died June 5, 1898. Frost's earlier schooling was at Beverley, whence in 1833 he went to Oakham School, remaining until October, 1835, when he entered St. John's College, Cambridge.

The Senior and Second Wrangler in 1839 were both elected to fellowships in their College on the same day, March 18th, 1839.

Frost illustrated another Cambridge peculiarity, great men choosing as a career to Tutor to private pupils. for example Hopkins, Frost, Routh. It is

DR. PERCIVAL FROST.

especially mentioned in the notice of him written for the Royal Society by his friend H. M. Taylor that his great success in obtaining private pupils when he returned to Cambridge in the Long Vacation succeeding his graduation induced him to abandon all idea of the legal profession, though urged by friends to read for the Bar, which indeed he had actually commenced to do. So settled and confident was he in this Cambridge profession of Private Tutor, that in 1841 he vacated his fellowship to marry Jennett Louise Dixon, of Oak Lodge, Finchley, with whom he "lived happy ever after" for 57 years !

Frost held a mathematical lectureship from 1847 to 1859 in Jesus College, from 1859 to 1889 in King's College ; but his chief work still consisted in the tuition of private pupils. As Frost himself was pupil of a Second Wrangler, Dr. John Hymers (1826), so his own greatest pupil was a Second Wrangler, W. K. Clifford, in 1867.

Frost edited Newton's Principia, Book I, sections 1—3, with notes, illustrations, and a collection of problems. First published in 1854, new editions appeared in 1863, 1878, and 1883. In 1863, with Joseph Wolstenholme, the noted problem maker, Frost published 'A Treatise on Solid Geometry.' When it was to be reissued, Wolstenholme withdrew, and the second edition 1875 and third edition 1886 were published by Frost alone, as also 'Hints for Solution of Problems in the Third Edition of Solid Geometry' in 1887. On this essential subject, this is one of the two great standard works in English.

In 1872 he published his famous 'Treatise on Curve-tracing.' In this treatise he presumed on the part of the reader no knowledge of the Differential Calculus, and restricted his field in other wise ways, until he humorously says, "In cutting off so many vital parts of a complete treatise I have to shew that I do not fall to the ground by sawing on the wrong side the branch on which I am sitting." In using the device of the Analytical Triangle, adopting Cramer's method of representing the possible terms by points, Frost was the first one to regard them merely as points referred to the sides of the triangle as coördinate axes, instead of regarding them with Cramer as marking the centers of the squares in which, in Newton's parallelogram, the values of the terms were to be inscribed.

This treatise cannot be too highly praised, and is still likely long to remain the greatest on the subject.

More than twenty papers by Frost on Algebra, Analytic Geometry, Lunar and Planetary Theories, and Electricity and Magnetism are mentioned in the Royal Society's 'Catalogue of Scientific Papers.'

Frost was made a Fellow of the Royal Society in 1882, and the same year was elected by King's College, Cambridge, to a terminable Fellowship, to which he was re-elected three times, holding it at his death. Like Sylvester, Frost was devoted to music with a fine execution on the piano and a penetrating appreciation of the masters.

Think of a man living 80 years scarcely knowing a day's illness ! This seems to an American as if he had never really tried his powers to the utmost.

Frost was, as R. Tucker writes, a great favorite at Cambridge, a sure concomitant of his brightness, cheerfulness, kindness of heart and consideration for others.

His exceptional life-long experience gives tremendous weight to his testimony in favor of the early cultivation of a talent for mathematics.

He explicitly says : "To attempt after a certain age to acquire ease in mathematical operations is like a grown man trying to learn the violin."

SOME REMINISCENCES IN REGARD TO SOPHUS LIE.

By DR. G. A. MILLER.

Since the death of the famous Norwegian mathematician, Sophus Lie, a large number of appreciative articles on his life and work have appeared in the mathematical journals of various countries. Two such articles were published in the April number of this journal. In what follows we shall endeavor to add a few details (based upon personal observations) which may assist those who never met him to form a more accurate idea with respect to him.

The trait of Lie's character which impressed me most forcibly when I first met him in the summer of 1895 was his extreme openness and lack of effort to hide ignorance on any subject. It is well known that he began the study of discontinuous groups under the masterly guidance of Sylow. He maintained a deep interest in these groups, making frequent use of Jordan's classical "Traité des Substitutions," and he spoke in terms of great respect of the work of Jordan, Frobenius, and others who were working in the field of discontinuous groups.

He, however, never felt at home in this subject. In fact he frequently remarked during his lectures that he always got stuck when he entered upon the subject of discontinuous groups, To him the continuous groups seemed more simple as well as more useful. He frequently used the expression, "the disconuous groups are good but my (continuous) groups are better," and he advised his students to begin their study of groups with the continuous groups and to take up the study of the discontinuous groups later.

He was an inspiring teacher but his lectures were not always well prepared. Sometimes he had to pay quite heavily for this lack of preparation, being unable at the moment to prove simple things in his own theory. It was an interesting sight to see him at the board working away with all his might and calling on his students to help him out of the difficulty, using the expression, "Here I stick, will not one of you help me out." He kept in good spirits at such occasions but he generally could not maintain enough self-possession to work his way out of the difficulties during the rest of the hour.

He was somewhat careless in regard to his dress. If he felt uncomfort-

able he would take off his collar and tie before his class, lay them aside, and then go on with his lecture. At one time the room happened to be somewhat cold so he put on his big hat and kept in on during the entire lecture. He considered his lectures as gymnastics for the students, being bent on training them to think according to his methods rather than on giving them a systematic treatise on any subject.

In his intercourse with the students he was friendly and approachable. He seemed to pay special attention to foreign students, claiming that they were generally better prepared than his German students. He would talk for hours with foreign students, asking questions in regard to their work as well as in regard to the mathematicians of their country. In case a student was well acquainted with any mathematical subject he would ask questions in regard to details which were unknown to him, candidly stating that he was asking for information.

During the summers of 1895 and 1896 he returned from his summer vacation about one month before the opening of the university and lectured daily before the few American students of mathematics who happened to be in Leipzig, with a view to prepare them to follow his lectures better during the university year. He did this work gratis and at his own suggestion. These lectures were very elementary and frequently took the form of a colloquium, the aim being to make the student familiar with the mathematical phrases rather than to teach much mathematics.

An idea of his view of mathematics in general may be inferred from the following incident. The building in which he held most of his lectures was royal property and the king of Saxony lived in a part of it whenever he came to Leipzig. One day Lie remarked to a few Americans, "Perhaps you do not know that our rooms are in the royal palace. It seems to me very appropriate that mathematics should be taught in a royal palace for it certainly is a royal subject. In some of the other departments one generation discards what the preceding generation regarded as established facts, but in mathematics an established fact remains valuable forever. We have just as much admiration for mathematical facts that were proved 2000 years ago as for those which have been established during our generation.

"Mathematics is a royal subject but the trouble is that the mathematicians do not get royal pay. Some of the professors here who lecture on popular subjects get several thousand dollars per year in students' fees, while the mathematician gets scarcely anything beyond his regular salary. In the popular departments the advanced subjects are frequently assigned to the younger men while the professors lecture on the elementary subjects in order to increase their income. In mathematics we are not tempted to commit this sin since all of our classes are small."

Lie was regarded as one of the easiest men at Leipzig for the doctor thesis. He generally assigned easy subjects and he assisted the students very freely. He was a hard worker, working seven days per week. He used to say

that he was almost certain to make mistakes in going over a piece of work for the first time, and that he could generally not find his errors until he would work over the problem a second time using different symbols. He frequently got very enthusiastic over the theory of groups remarking (not very seriously) that everything could be done by means of groups.

The students talked a great deal about his peculiarities but they had great respect for his attainments. In speaking about Klein and Lie, a Frenchman who had studied under both remarked, "Klein is a gentleman while Lie is just a good fellow." I presume many of those who studied under both would agree that this statement conveys a great deal of truth.

I feel encouraged to publish these few personal observations because I believe that Lie would have liked to have his faults go with his merits. I do not think he wanted to be regarded as one who thought that he had mastered every part of the extensive science of mathematics or as one who thought that all his habits were exemplary. He believed that he had contributed materially towards the advancement of the science of mathematics and he worked hard to accomplish this end. Future developments along the lines which have been emphasized or opened by him will have a great influence on his relative position among the mathematicians of this century.

Cornell University, June, 1899.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

Grassmann's *Ausdehnungslehre* is one of the few great works of mathematics of the 19th century. Appearing first in 1844 and rewritten in 1862, it is only within the last decade or two that it has received a tardy recognition. One reason for this is found in the difficulty of the subject itself, being unlike other mathematics ; and another, in the rigorous methods of presentation adopted by the author. In the *Ausdehnungslehre* of 1862, following some 150 pages of theory, the author for the first time gives the subject concrete form by applying his method to geometry. The theoretical part is naturally the more difficult, while the application to geometry is the more interesting. Hyde, in his *Directional Calculus*, purposing to present the *Ausdehnungslehre* to American readers, cut the knot of the difficulty by taking the results of the theoretical part for granted and giving only the application to geometry, and by limiting his treatment to two and three dimensional space.

An elementary exposition which will give the simpler portions of the theoretical part as well as the applications of the theory seems to be needed.

Such an exposition should serve the needs of two classes of readers : First, of those who would like to have a good general idea of the subject without going very deeply into its particulars ; and secondly, of those who, expecting to make a thorough study of the subject, wish first to read an introduction to it. To meet this want the following pages have been written. In some places for the purpose of making the subject clearer, changes have been made, but wherever they have been introduced, attention is always called to them.

CHAPTER I.

INTRODUCTION.

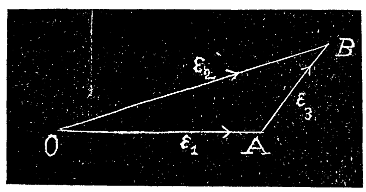
1. In elementary mathematics only one kind of unit is admitted, or at most two, viz., 1 and $\sqrt{-1}$. In the Theory of Extension besides the absolute unit of arithmetic and algebra, *extensive quantities* appear. The extensive quantities are different in nature from the absolute unit and different from each other. As a simple example of an extensive quantity we may name a *vector* (§3).

2. A *Scalar* quantity is a quantity of elementary mathematics, *i. e.*, a simple number, either positive or negative.

3. *A Vector* is a straight line whose length and direction are fixed but not its position. Thus any two parallel and equal straight lines may represent the same vector. A vector gives the relative position of one point with reference to another, viz., a certain distance in a certain direction.

4. Vectors can be *added* and *subtracted.*

Thus if ε_1 is the vector from O to A and ε_2 the vector from A to B, the sum of ε_1 and ε_2 is ε_3, because translation from O to B along the straight line OB is equivalent, or equal in the vector sense, to translation along OA and AB. Thus,



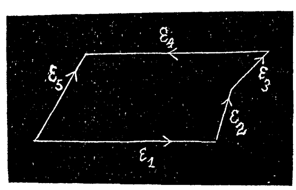
$$\varepsilon_1+\varepsilon_2=\varepsilon_3.$$

Transposing, we get

$$\varepsilon_1=\varepsilon_3-\varepsilon_2=\varepsilon_3+(-\varepsilon_2).$$

Interpreting this equation we see that translation along ε_3 followed by translation along ε_2 *in the negative direction* is equal to translation along ε_1.

The sum of any number of vectors may evidently be found in the same way. Thus, in the figure

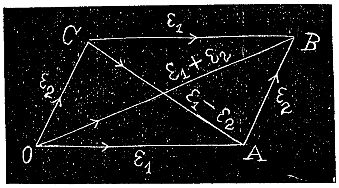


$$\varepsilon_5=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4.$$

Stated generally, we have

The sum of any number of vectors is found by joining the beginning point of the second vector to the end point of the first, the beginning point of the third to the

end point of the second, and so on ; the vector from the beginning point of the first vector to the end point of the last is the sum required.



The sum and difference of two vectors are the diagonals of the parallelogram whose adjacent sides are the given vectors.

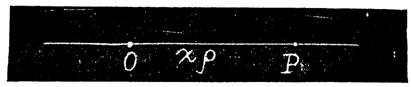
Or, more explicitly—

(1). *The sum of two vectors going out from an origin and forming two sides of a parellelogram is that diagonal of the parallelogram which passes through the origin.*

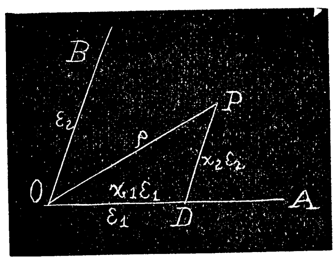
(2). *The difference of two such vectors is that diagonal which proceeds from the end of the subtrahend vector to the end of the minuend vector.*

5. Vectors and line segments give us simple examples of extensive quantities. We proceed to show how lines and vectors can be used in a system of coördinates.

6. The simplest case of this is where a point is located on a given line. Let ρ denote any given line, and let O be an origin on it. Let further x be a scalar. Then by giving the proper value to x, $x\rho$ will locate any point P whatever on the line. Here ρ is to be regarded as an extensive quantity since it denotes not a number but the position and length of a line.



7. The next simplest case of coördinates is that in which a point is located in a plane by means of two vectors.



Let O, the origin, be a point in the given plane, and ε_1 and ε_2 two unit vectors in this plane. Then, by making

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2$$

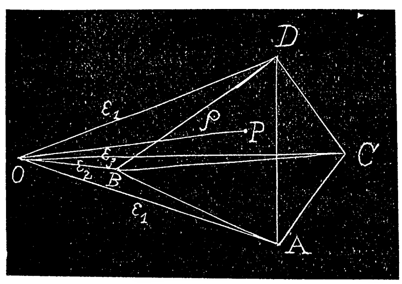
$$\text{where } x_1 = \frac{r\sin BOP}{\sin BOA}, \quad x_2 = \frac{r\sin POA}{\sin BOA},$$

and r=length of ρ, the point P may be located at any point of the plane. Here ε_1, ε_2 and ρ are extensive quantities, and x_1 and x_2 are scalars.

8. In space we may have a similar system containing three vectors. Thus, if

$$\rho = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$$

by assigning values to x_1, x_2, and x_3, P the extremity of ρ may be located at any point in space.



9. In the same way we can have a system including four vectors. Thus, if P is any point, $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ are four unit vectors, and

$$x_1=\frac{\text{pyramid } P-BDC}{\text{pyramid } A-BDC}, \quad x_2=\frac{\text{pyramid } P-ADC}{\text{pyramid } B-ADC},$$

etc., we have the equation

$$\rho=x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+x_4\varepsilon_4,$$

i. e. translation along ε_1 a distance equal to x_1, followed by translation along ε_2 a distance equal to x_2, and so on, is equivalent to translation from O to P direct. A geometrical proof of the truth of this can be given, but it is not thought necessary to insert it here.

REMARK. In the preceding the ε's in a certain sense denote dimensions. Then the *space* considered in this last article is of the *fourth* order.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF EXTENSIVE QUANTITIES.

10. DEFINITION. A quantity is said to be *independent* when it can not be expressed in terms of others. A quantity is said to be *dependent* when it can be numerically expressed in terms of others, *i. e.* as a sum formed out of numerical multiples of these quantities.

Thus if a_1, a_2, ... are extensive quantities, and α_1, α_2, ... are real numbers, positive or negative, and

$$a=\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots$$

a is said to be a dependent quantity and to be '*numerically expressed*' in terms of a_1, a_2, a_3, ...

11. A quantity a_1 is a *Unit* if it can serve along with other like units, a_2, a_3, ... to give a series of numerically derived quantities, a...... A unit is said to be *original* if it is not derived from other units. A set of quantities which are independent, *i. e.*, no one of which is numerically expressible in terms of one or more of the others is called a *System of Units*, provided any number of other quantities can be expressed in terms of them.

As an illustration of such a system we may take the set of units given in Art. 8. There ε_1, ε_2, ε_3 are a set of quantities which are independent because any sum formed out of multiples of ε_1 and ε_2 can never be a quantity like ε_3, since any sum formed from ε_1 and ε_2 would be a quantity in the plane of these two (7) while ε_3 is outside of this plane. Moreover, any number of other quantities, ρ's, can be derived from ε_1, ε_2, ε_3.

12. DEFINITION. An Extensive *Quantity* is a quantity numerically derived from a *system* of units. If an extensive quantity can be derived from the original units it is called an extensive quantity of the *first kind*.

13. DEFINITION. Quantities from the same system can be added (or subtracted) by adding the numerical coefficients of the same units.

Thus

$$\Sigma \alpha e + \Sigma \beta e = \Sigma(\alpha+\beta)e$$

where the α's and β's under the summation signs are numbers, and the e's are extensive units. We may add here that in the *Ausdehnungslehre* the distributive law is always assumed to hold.

REMARK. To remove ambiguity it will be understood that all indicated operations are performed as one comes to them from the left. Thus $a+b+c$ means $(a+b)+c$, and abc means $(ab)c$.

14. The following formulas underlie and justify all the operations involved in addition and subtraction in algebra. They follow directly from the definition in 13.

(1). $a+b=b+a$, a commutative law in addition and subtraction.

(2). $a+(b+c)=a+b+c$, associative law in addition and subtraction.

$\left.\begin{array}{l}(3).\ a+b-b=a\\(4).\ a-b+b=a\end{array}\right\}$ *opposite* character of addition and subtraction.

Hence all the laws for addition and subtraction of algebraic numbers hold also for extensive quantities.

15. DEFINITION. When an extensive quantity is multiplied (or divided) by a *number* each of its coefficients is multiplied by that number.

Thus

$$\Sigma \alpha e . \beta = \Sigma(\alpha\beta)e.$$

REMARK. If a is an extensive quantity and α a number, then in αa or $a\alpha$ the numerical factor is the multiplier and the other factor is the multiplicand.

16. From Art. 15 we infer the following formulas :

(1). $a\alpha=\alpha a$,

(2). $a\beta\gamma=a(\beta\gamma)$,

(3). $(a+b)\gamma=a\gamma+b\gamma$,

(4). $a(\beta+\gamma)=a\beta+a\gamma$,

where, as heretofore, the Greek letters denote real numbers and the Roman letters, extensive quantities. From these formulas it follows—

That all the laws of multiplication and division of algebraic quantities hold also for extensive quantities multiplied or divided by numbers.

17. DEFINITION. The totality of quantities which are derivable from a series of extensive quantities, a_1, a_2, a_3, ... a_n is called the *Space* of those quantities. A space which can be formed out of not less than n such quantities each of the first kind (12) is called a space of the nth order.

18. DEFINITION. If every quantity of a space (A) is at the same time a quantity of another space (B), while the converse is not true, then the spaces are said to be *incident;* the first is said to be *subordinate* to the second, and the second, to *include* the first.

19. *If n independent quantities a_1 ..... a_n can be numerically expressed in*

terms of n other quantities $b_1 \ldots\ldots b_n$, *then is the space of the first quantities identical with that of the last quantities. But if the n quantities* $a_1 \ldots\ldots a_n$ *can be expressed in terms of less than n quantities* $b_1 \ldots\ldots b_n$, *then* $a_1 \ldots\ldots a_n$ *are not independent, and some of them can be numerically expressed in terms of others.*

20. *Two quantities of a space of the nth order are equal to each other when and only when their numerical coefficients of the same units are equal.* This is analogous to the algebraic theorem which says that two complex numbers are equal only when their real parts are equal and also their imaginary parts.

21. *If the coefficients* $x_1 \ldots\ldots x_n$ *by which an extensive quantity* x *is expressed in terms of the units* $e_1 \ldots\ldots e_n$ *satisfy an equation of the mth degree* $f(x_1 \ldots\ldots x_n)=0$, *then the coefficients* $y_1 \ldots\ldots y_n$ *by which* x *is expressed in terms of* $a_1 \ldots\ldots a_n$ *of the same space also satisfy an equation of the mth degree, and if the first equation is homogeneous, the latter is also.*

Proof. Let $a_1 = \Sigma\alpha_{1r}e_r$, ... Then we have

$$x_1e_1+x_2e_2+\ldots\ldots+y_1\Sigma\alpha_{1r}e_r+y_2\Sigma\alpha_{2r}e_r+\ldots\ldots=\Sigma y_r\alpha_{r1}.e_1+\Sigma y_r\alpha_{r2}.e_2+\ldots\ldots$$

$$\therefore\ x_1=\Sigma y_r\alpha_{r1},\ x_2=\Sigma y_r\alpha_{r2},\ \ldots\ldots\quad \text{(Art. 20).}$$

But if these values are substituted in $f(x_1, \ldots\ldots x_n)=0$, we get an equation of the mth degree in y_1, y_2, ..., and, indeed, homogeneous if the first equation is homogeneous.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\dfrac{.\dot{2}9\dot{7}}{1.00\dot{2}}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain %?

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville Tenn., and the PROPOSER.

$.\dot{2}9\dot{7}=\frac{297}{999}=\frac{11}{37}$; $1.00\dot{3}=1\frac{1}{300}=\frac{301}{300}$.

$\frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}$; $.\frac{3300}{11137}=\frac{330}{11137}$.

$\therefore\ \$4.\dfrac{\dot{2}9\dot{7}}{1.00\dot{3}}=\$4\frac{330}{11137}=\$\frac{44878}{11137}=$ cost price.

$\$6.\frac{1000}{33337}=\$6\frac{100}{33337}=\$\frac{200122}{33337}=$ selling price.

$\therefore\ \$\frac{200122}{33337}-\$\frac{44878}{11137}=\$\frac{19801644}{100344437}=$ gain.

$(\frac{19801644}{100344437}\div\frac{44878}{11137})$ of $100\%=48\frac{19640328}{20217539}\%$.

Also solved by *O. S. WESTCOTT* and *ALOIS F. KOVARIK*.

113. Proposed by B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.

In what time can a note of \$5280, bearing 6% interest, be paid by paying \$600 a year? [Solve by arithmetic].

Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Let Q be the principal, r the rate, n the number of years, and P annual payment. Then

$Q(1+r)=$ amount due at end of first year.

$Q(1+r)-P=$ principal to run second year.

$Q(1+r)^2-P(1+r)=$ amount due at end of second year.

$Q(1+r)^2-P(1+r)-P=$ principal to run third year.

....

$Q(1+r)^n-P(1+r)^{n-1}\ldots\ldots P(1+r)-P=$ amount to run $(n+1)$th year.

But the debt is cancelled. Hence

$$Q(1+r)^n-P(1+r)^{n-1}-\ldots\ldots P(1+r)-P=0.$$

$$\therefore P\left[\frac{(1+r)^n-1}{r}\right]=Q(1+r)^n.$$

$$\therefore (P-Qr)(1+r)^n=P.$$

$$\therefore\ n=\frac{\log P-\log(P-Qr)}{\log(1+r)}.$$

In the problem $P=\$600$, $Q=\$5280$, $r=.06$.

$$\therefore\ n=\frac{\log 600-\log(600-5280\times.06)}{\log(1.06)}=12.88 \text{ years.}$$

Also solved by *G. B. M. ZERR*, *COOPER D. SCHMITT*, and *J. SCHEFFER*.

ALGEBRA.

93. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Given $x^x+y^y=285$, and $y^x-x^y=14$, to find the values of x and y. [From *Bonnycastle's Algebra*, 1841].

I. Solution by A. H. BELL, Hillsboro, Ill.

The two equations give

$$(285-y^y)^y=(y^x-14)^x\ldots\ldots(1).$$

In (1), assuming y equal to 3, 2.9, and 2.8, by logarithms, $y=2.8248$ nearly, $=\log 0.450987$.

Multiplying by y, $\log(y^y)=\log 1.273952,=18.791=y^y$.

But $x^x=285-y^y$.

$\therefore\ 285-18.791=266.791,=x^x$.

In a table of logarithms by $\log x+\log.$ of $\log x$, x is found to be nearly $4.01637+$.

II. Solution by J. M. BOORMAN, Consultative Mechanician, Etc., Woodmere, N. Y.

$x=4.016631(8\pm;\ x_1=4.04444(4\pm;\ x_2=4.045\pm;^*\ x_a<0=-?$;

$y=2.8157(12\pm;\ y_1=-1.922(77\pm;\ y_2=-1.92\pm;\ y_a>0=+?$;

curiously enough always a quartic (even if $x=6$ or more, $y=5$ or more).

*Because $x=v$ nearly in $v^v=285$, as if $y<0$. $\therefore\ y^y=(1/y)^n$; $x^y=(1/x)^n=$ fractions. $\therefore$ as $v=4.0449$ nearly, x_1 (must) <4.0449, x_2 (must) >4.0449, so $y_2{}^{x_2}$ (must)$=14.069$ nearly by (2), as $x_2{}^{y_2}=$a negative fraction$=0.069\pm$. Solve $10^{x\log x}+10^{y\log y}=10^{\log 285}$......(1), and $10^{x\log y}-10^{y\log x}=10^{\log 14}$ .....(2), by double position. Note that $10^x=10^{4016631}=4016631$ power,—four millionth, etc., power of the millionth root of 10. So, curiously, we must compute to an even decimal if $x>0$, and to an odd decimal digit if $x<0$, to get true results.

III. Solution by OLIVER S. WESTCOTT, Principal North Division High School, Chicago, Ill.

I know of no direct method of solution, and as *Bonnycastle's Algebra*, from which it was taken, gives some idea of the use of logarithms and also of the rule known to arithmeticians in the days of Daboll as Double Position, it is presumable at least, that Bonnycastle's method was something like the following:

If x were 4 and y were 3, we should have $4^4+3^3=283$ and $3^4-4^3=17$.

Evidently then, $x>4$ and $y<3$.

Put $y=2.75$. Then $x^x+2.75^{2.75}=285$, and $x^x=268.85028$.

$x\log x=\log 268.85028$, $x=4.0205$.

Substituting these values of x and y in equation (2) we have

$(2.75)^{4.0205}-(4.0205)^{2.75}=12.494296$. $14-12.494296=1.50570$ error.

Again, put $y=2.85$. Then $x^x+(2.85)^{2.85}=285$. $x^x=265.21628$. $x\log x=\log 265.21628$. $x=4.0148$.

Substituting as before,

$(2.85)^{4.0148}-(4.0148)^{2.85}=14.471434$. $14.471434-14=0.471434$ error,

$$\text{and } \frac{0.471434\times 2.75+1.505704\times 2.85}{1.505704+0.471434}=2.82615$$

for an approximate value of y.

Since 2.85 is plainly much nearer the result sought for than 2.75, put $y=2.83$.

Then $x^x+(2.83)^{2.83}=285$. $x^x=266.00867$. $x\log x=\log 266.00867$. $x=4.01605$.

Substituting in equation (2) as before $(2.83)^{4.01605}-(4.01605)^{2.83}=14.083487$.

$14.083487-14=0.083487$ error. Then

$$\frac{0.471434\times2.83-0.083487\times2.85}{0.471438-0.083487}=2.825696.$$

Bonnycastle's answer for y is 2.825716.

Again $x^x+(2.825696)^{2.285696}=285$. $x^x=266.17636274$. $x\log x=\log 266.17636274$. $x=4\ 016348$.

Bonnycastle's answer for x is 4.016698.

These trifling discrepancies may easily be occasioned by variation in the logarithmic tables used. I have made use of Chambers's seven place tables.

I have not succeeded in effecting a neat and satisfactory solution ; by trial, however, I find, approximately, $x=3.89$, $y=3.54$. G. B. M. Zerr.

I do not believe a strictly algebraic solution of this problem can be given, because it is not an algebraic, but a transcendental function. Algebraically there are really four unknowns, x, $\log x$, y, and $\log y$. Of course, $\log x$ and $\log y$ are known if x and y are known, but they are here algebraically separate unknowns because of the manner in which they enter the equations. W. H. Drane.

GEOMETRY.

114. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If a variable ellipse hyperosculate a fixed ellipse at the extremity of the minor axis, the locus of the foci is a circle whose diameter is equal to the radius of curvature.

Solution by the PROPOSER.

Let $x^2/a^2+y^2/b^2=1$......(1) be the given ellipse ; then the tangent at the vertex of the minor axis is $y/b-1=0$......(2), and the variable conic is

$$x^2/a^2+y^2/b^2-1-\lambda[(y/b)-1]^2=0\text{......(3), or}$$

$$x^2/a^2+(1-\lambda)(y^2/b^2)+(2\lambda/b)y-(\lambda+1)=0\text{ .....(4).}$$

Now, the foci of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0\text{......(5),}$$

are given by

$$C(x^2-y^2)+2Fy-2Gx+A-B=0\text{......(6),}$$

$$Cxy-Fx-Gy+H=0\ldots\ldots(7),$$

in which $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$, $F=gh-af$, $G=hf-bg$, $H=fg-ch$.

For the conic (4), $A=-1/b^2$, $B=-[(1+\lambda)/a^2]$, $C=[(1-\lambda)/(a^2b^2)]$, $F=-\lambda/a^2b^2$, $G=H=0$, and these in (6) and (7) give

$$(1-\lambda)(x^2-y^2)-2b\lambda y+b^2(1-\lambda)-a^2=0\ldots\ldots(8),$$

$$(1-\lambda)y+b\lambda=0\ldots\ldots(9).$$

Eliminating λ from (8) and (9), the required locus is

$$x^2+y^2+\frac{a^2-2b^2}{b}y=a^2-b^2\quad\ldots\ldots(10).$$

This is a circle, radius $a^2/2b$, or one-half the radius of curvature of (1) at the extremity of the minor axis.

115. Proposed by **MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.**

The locus of a point such that the sum of the squares of its normals form a given ellipsoid is constant, is a co-axial ellipsoid. [From *C. Smith's Solid Analytical Geometry*, page 95.]

Solution by **G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let (l, m, n) be the point whose locus is required. Then from the conditions of the problem we have

$$(l-x_1)^2+(l-x_2)^2+(l-x_3)^2+(l-x_4)^2+(l-x_5)^2+(l-x_6)^2$$
$$+(m-y_1)^2+(m-y_2)^2+(m-y_3)^2+(m-y_4)^2+(m-y_5)^2+(m-y_6)^2$$
$$+(n-z_1)^2+(n-z_2)^2+(n-z_3)^2+(n-z_4)^2+(n-z_5)^2+(n-z_6)^2=C^2.$$

This equation can be written more briefly thus :

$$\Sigma(l-x_2)^2+\Sigma(m-y_1)^2+\Sigma(n-z_1)^2=C.$$

$$\therefore\ C(l^2+m^2+n^2)-2[l\Sigma(x_1)+m\Sigma(y_1)+n\Sigma(z_1)]$$
$$+\Sigma(x_1^2)+\Sigma(y_1^2)+\Sigma(z_1^2)=C^2\ldots\ldots(1).$$

The equation of the normal is

$$(a^2/x)(l-x)=(b^2/y)(m-y)=(c^2/z)(n-z)\ldots\ldots(2).$$

Also, $x^2/a^2+y^2/b^2+z^2/c^2=1\ldots\ldots(3)$.

The values of y, z in terms of x from (2) in (3) gives

$$x^2/a^2+\frac{b^2m^2x^2}{(a^2l-a^2x+b^2x)^2}+\frac{c^2n^2x^2}{(a^2l-a^2x+c^2x)^2}=1\ldots\ldots(4).$$

(4) can be written as follows :

$$x^6+Ax^5+Bx^4+Dx^3+Ex^2+Fx+G=0.$$

Then $\Sigma(x_1)=-A$, $\Sigma(x_1x_2)=B$, $\Sigma(x_1^2)=A^2-2B$.

But $A=-\dfrac{2a^2l}{a^2-c^2}-\dfrac{2a^2l}{a^2-b^2}$.

$$B=\frac{a^2(a^2l^2+c^2n^2)}{(a^2-c^2)^2}+\frac{a^2(a^2l^2+b^2m^2)}{(a^2-b^2)^2}+\frac{4a^4l^2}{(a^2-b^2)(a^2-c^2)}-a^2.$$

$$\therefore \Sigma(x_1)=\frac{2a^2l}{a^2-c^2}+\frac{2a^2l}{a^2-b^2}.$$

$$\Sigma(x_1^2)=\frac{2a^2(a^2l^2-c^2n^2)}{(a^2-c^2)^2}+\frac{2a^2(a^2l^2-b^2m^2)}{(a^2-b^2)^2}+2a^2.$$

By symmetry,

$$\Sigma(y_1)=\frac{2b^2m}{b^2-c^2}+\frac{2b^2m}{b^2-a^2}.$$

$$\Sigma(y_1^2)=\frac{2b^2(b^2m^2-c^2n^2)}{(b^2-c^2)^2}+\frac{2b^2(b^2m^2-a^2l^2)}{(b^2-a^2)^2}+2b^2.$$

$$\Sigma(z_1)=\frac{2c^2n}{c^2-b^2}+\frac{2c^2n}{c^2-a^2}.$$

$$\Sigma(z_1^2)=\frac{2c^2(c^2n^2-b^2m^2)}{(c^2-b^2)^2}+\frac{2c^2(c^2n^2-a^2l^2)}{(c^2-a^2)^2}+2c^2.$$

Substituting these values in (1) and reducing we get

$$6(l^2+m^2+n^2)+\frac{2(c^2n^2-a^2l^2)}{a^2-c^2}+\frac{2(b^2m^2-a^2l^2)}{a^2-b^2}+\frac{2(c^2n^2-b^2m^2)}{b^2-c^2}$$
$$=C^2-2(a^2+b^2+c^2).$$

$$\therefore \frac{2(a^4-2a^2b^2-2a^2c^2+3b^2c^2)l^2}{(a^2-c^2)(a^2-b^2)(C^2-2a^2-2b^2-2c^2)}$$

$$+\frac{2(2b^2c^2+2a^2b^2-3a^2c^2-b^4)m^2}{(a^2-b^2)(b^2-c^2)(C^2-2a^2-2b^2-2c^2)}$$

$$+\frac{2(c^4-2a^2c^2-2b^2c^2+3a^2b^2)n^2}{(a^2-c^2)(b^2-c^2)(C^2-2a^2-2b^2-2c^2)}=1.$$

$\therefore\ l^2/R^2+m^2/S^2+n^2/T^2=1$, a co-axial ellipsoid.

CALCULUS.

89. Proposed by **WILLIAM HOOVER, A. M., Ph. D.,** Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Integrate the equation, $\dfrac{dy}{dx}+y\cos x=\dfrac{\sin 2x}{2}$.

I. Solution by **Dr. E. D. ROE, Jr.,** Norwood, Mass.

Put $\sin x-y=z$; then $\cos x dx-dy=dz$, $dy=\cos x dx-dz$, and the equation becomes

$$\frac{dz}{dx}+(z-1)\cos x=0, \text{ or } \frac{dz}{z-1}+\cos x dx=0.$$

Integrating this, $\log(z-1)+\sin x+\kappa=0$, $z-1=e^{-\sin x-\kappa}$,

$$\text{or } \sin x-y-1=e^{-\sin x-\kappa}=-ce^{-\sin x},\ y=\sin x-1+ce^{-\sin x}$$

Dr. Roe also furnished a second solution.

II. Solution by **F. ANDEREGG, A. M.,** Professor of Mathematics, Oberlin College, Oberlin, O.

If the equation $\dfrac{dy}{dx}+y\cos x=0$ is solved, the result is $y=C_1e^{-\sin x}$. After substituting in the original equation $\dfrac{dC_1}{dx}$ is found to equal $e^{\sin x}\sin x\cos x$; therefore, $C_1=e^{\sin x}\sin x-e^{\sin x}+C$. And $y=\sin x+Ce^{-\sin x}$.

III. Solution by **WALTER H. DRANE,** Graduate Student, Harvard University, Cambridge, Mass., and **JOHN R. JEFFERY,** Student in Ohio State University, Columbus, O.

The general form of this equation is $\dfrac{dy}{dx}+Py=Q$ of which the general integral is $e^{\int Pdx}y=\int e^{\int Pdx}Qdx+C$. Here $\int Pdx=\int\cos x dx=\sin x$.

$\therefore\ e^{\sin x}y=\int e^{\sin x}\cos x\sin x dx+C$.

Integrating right member by parts, $e^{\sin x}y=\sin x.e^{\sin x}-e^{\sin x}+c$, or $y=\sin x-1+ce^{-\sin x}$.

[See Johnson's *Differential Equations*, page 35, ex. 7.]

IV. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pa.; M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Ind.; and BURKE SMITH, Senior Class, University of Washington, Seattle, Wash.

$\dfrac{dy}{dx}+y\cos x=\sin x\cos x.$

Multiply by $e^{\sin x}$ as an integrating factor.

Then $e^{\sin x}dy+y\cos x.e^{\sin x}dx=e^{\sin x}\sin x\cos x dx.$

Integrating, $y.e^{\sin x}=\int e^{\sin x}\sin x\cos x dx.$

Integrate right hand member by parts, and we have,

$$ye^{\sin x}=e^{\sin x}\sin x-\int e^{\sin x}\cos x dx=e^{\sin x}\sin x-e^{\sin x}+c.$$

$\therefore\ y=\sin x-1+ce^{-\sin x}.$

V. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $\sin x=\log v$; so that $\cos x dx=dv/v$. Then the equation becomes $dy/dv+y/v=\log v/v$.

$\therefore\ vdy+ydv=\log v dv.\quad \therefore\ vy=v(\log v-1)+C.\quad \therefore\ y+1=\log v+C/v.$

$\therefore\ y+1=\sin x+Ce^{-\sin x}.$

VI. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and ARTHUR C. LUNN, University of Chicago, 5630 Drexel Ave., Chicago, Ill.

Solving first the differential equation $\dfrac{\partial y}{\partial x}+y\cos x=0$, we find $y=Ce^{-\sin x}$; differentiating, we have $\dfrac{\partial y}{\partial x}=-Ce^{-\sin x}\cos x+e^{-\sin x}\dfrac{\partial C}{\partial x}$.

$$\therefore\ e^{-\sin x}\frac{\partial C}{\partial x}=\tfrac{1}{2}\sin 2x.\quad \therefore\ C=\tfrac{1}{2}\int e^{\sin x}\sin 2x.dx.$$

Putting $\sin x=z$, $\therefore\ \cos x=\sqrt{1-z^2}$, $\partial x=\dfrac{\partial z}{\sqrt{1-z^2}}$; we have,

$$C=\int e^z.z\partial z=\int z\partial(e^z)=ze^z-\int e^z\partial z=ze^z-e^z=\sin x.e^{\sin x}-e^{\sin x}+c.$$

Substituting this in $y=Ce^{-\sin x}$, we obtain $y=\sin x-1+ce$.

The method employed is Lagrange's method of variation of parameters.

Also solved by *HENRY HEATON*, and *P. H. PHILBRICK*.

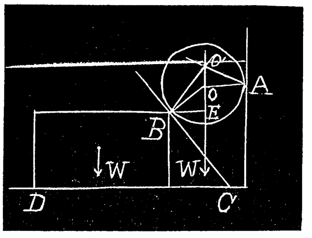
MECHANICS.

82. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A sphere, diameter $2a$. rests in limiting equilibrium upon the edge of a box and against a vertical wall. If the box be of such dimensions that it will not tip, find the distance of the box from the wall, having given the coefficient of friction between the sphere and wall $\frac{1}{2}$, between the sphere and box $\frac{1}{3}$, and between the box and floor $\frac{2}{3}$. [From Problems in Mechanics proposed to class in Harvard University.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let W=weight of sphere, W'=weight of box, $\mu=\frac{1}{2}$, $\mu'=\frac{1}{3}$, $\mu''=\frac{2}{3}$, $\theta=\angle BCD$, S=normal reaction of wall, R=normal reaction of box, d=distance of box from wall.



$\therefore\ d=AO+BE=a(1+\sin\theta)$, since BO is perpendicular to BC.

Also $S=\mu' R\cos\theta+R\sin\theta=\mu'' W'$ (resolving horizontally).

$$\therefore\ S=\tfrac{2}{3}W',\ R=\frac{\mu'' W'}{\mu'\cos\theta+\sin\theta}=\frac{2W'}{\cos\theta+3\sin\theta}.$$

Also $\mu S+\mu' R\sin\theta+R\cos\theta=W$ (resolving vertically), or $\frac{1}{2}S+\frac{1}{3}R\sin\theta+R\cos\theta=W$.

The values of S and R in the last equation give

$$\tfrac{1}{3}W'+\frac{2W'\sin\theta}{3\cos\theta+9\sin\theta}+\frac{2W'\cos\theta}{\cos\theta+3\sin\theta}=W.$$

$$\therefore\ \tan\theta=\frac{7W'-3W}{9W-5W'},\ \sin\theta=\frac{7W'-3W}{\sqrt{90W^2-132WW'+74W'^2}}.$$

$$\therefore\ d=a\left[\frac{\sqrt{90W^2-132WW'+74W'^2}+7W'-3W}{\sqrt{90W^2-132WW'+74W'^2}}\right].$$

If $W=W'$, $d=\frac{1}{2}a(2+\sqrt{2})$.

83. Proposed by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

A particle is projected upwards in vacuo with a velocity v. Show that on reaching the ground again there is no deviation to the south, but the deviation to the west is $4\omega\cos\lambda(v^3/3g^2)$. [*Laplace*, iv, page 341.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The equations of motion are

$$\frac{d^2x}{dt^2}+2\omega\sin\lambda\frac{dy}{dt}=0,\ \frac{d^2y}{dt^2}-2\omega\cos\lambda\frac{dz}{dt}-2\omega\sin\lambda\frac{dx}{dt}=0,\ \frac{d^2z}{dt^2}+2\omega\cos\lambda\frac{dy}{dt}=-g.$$

(Routh's *Advanced Rigid Dynamics*, fourth edition, page 20.)

As a first approximation, we can neglect the motion of the earth. Then, from mechanics, $x=0$, $y=0$, $z=vt-\frac{1}{2}gt^2$.

$$\therefore \frac{dx}{dt}=0,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=v-gt.$$

$$\therefore \frac{d^2x}{dt^2}=0,\ \frac{d^2y}{dt^2}=2\omega\cos\lambda(v-gt).$$

$\therefore$ $x=0$, $y=\omega\cos\lambda(vt^2-\frac{1}{3}gt^3)$. But $t=2v/g$.

$\therefore$ $x=0$, $y=\omega\cos\lambda(4v^3/g^2-8v^3/3g^2)=4\omega\cos\lambda(v^3/3g^2)$.

$\therefore$ deviation south$=0$, west$=4\omega\cos\lambda(v^3/3g^2)$.

AVERAGE AND PROBABILITY.

67. Proposed by HENRY HEATON, M. Sc., Atlantic, Ia.

A witness in court who undertook to recognize the signature of an individual failed four times in succession. What is the probability that he was correct the fifth time? An actual occurrence.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chaster High School, Chester, Pa., and CHARLES CARROLL CROSS, Libertytown, Md.

Let p=chance, p_1=chance of failure.

$$\therefore p_1=\frac{\int_0^1 x^5dx}{\int_0^1 x_4dx}=\tfrac{5}{6}.\quad \therefore p=1-\tfrac{5}{6}=\tfrac{1}{6}.$$

68. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

What are the odds against throwing 7 or 11 at one throw with two dice?

I. Solution by CHARLES CARROLL CROSS, Libertytown, Md.

Each dice may fall in any one of 6 ways, therefore, both dice in $6\times6=36$ ways.

$11=6+5$ or $5+6$; hence the chance against throwing 11 at one throw is $1-\frac{2}{36}=\frac{17}{18}$.

$7=1+6=2+5=3+4=4+3=5+2=6+1$; hence the chance against throwing 7 at one throw is $1-\frac{6}{36}=\frac{5}{6}$.

Hence, the chance against throwing either 7 or 11 is $1-(\frac{1}{18}+\frac{1}{6})=\frac{7}{9}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

The chance of throwing 7 with two dice is $\frac{6}{36}=\frac{1}{6}$, and that of throwing 11 is $\frac{2}{36}=\frac{1}{18}$; therefore, the chance of throwing either 7 or 11 is $\frac{1}{6}+\frac{1}{18}=\frac{2}{9}$.

$\therefore$ the odds against this event$=7:2$.

III. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

The combinations which give 7 or 11 with two dice are four in number, viz.: $6+1$, $5+2$, $4+3$, and $6+5$.

The total number of combinations is 36. Hence the chance of throwing either a 7 or 11 at one throw is $\frac{4}{36}=\frac{1}{9}$. Hence the odds are 8 to 1 against the event.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science. Chester High School, Chester, Pa.

7 can be thrown 6 ways at one throw with two dice, as follows:

6 and 1, 1 and 6, 5 and 2, 2 and 5, 4 and 3, 3 and 4.

$\therefore$ chance of throwing 7 is $\frac{6}{36}=\frac{1}{6}$.

$\therefore$ the odds are 5 to 1 against the event.

11 can be thrown 2 ways, as follows: 6 and 5, and 5 and 6.

$\therefore$ chance of throwing 11 is $\frac{2}{36}=\frac{1}{18}$.

$\therefore$ the odds are 17 to 1 against the event.

7 is the most likely throw of all at one time with two dice.

V. Solution by ELMER SCHUYLER, Annapolis, Md.

$7=1+6=2+5=3+4$, each occuring in two ways.

$\therefore$ 7 can occur in six ways.

$11=5+6$, can occur in 2 ways.

$\therefore$ 7 and 11 can occur in eight ways.

$\therefore \frac{8}{36}=\frac{2}{9}=$probability in favor of the event.

$\therefore$ the odds are 7 to 2.

69. Proposed by Rev. W. ALLEN WHITWORTH, M. A.

There are n equal sugar sticks. Each stick is broken into two pieces, all positions of the fracture being equally likely. Of the two n pieces thus formed, a child is to take the largest. Show that his expectation is $[2n+1]/[2(n+1)]$ of a stick. [From *The Educational Times*, June, 1898.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let each stick be of unit length. Then since there are $2n$ pieces, and each piece can have any length from zero to unity, we have for the required expectation:

$$E=\frac{\int_0^1 x^{2n+1}dx}{\int_0^1 x^{2n}dx}=\frac{2n+1}{2(n+1)}.$$

70. Proposed by Professor MILLER.

A ship at A observes another at B, whose course is unknown. Supposing their speed the same, prove that the chance of their coming within a given distance, d, of each other is always $(2/\pi)\sin^{-1}(d/a)$, whatever the course taken by A; provided its inclination to AB is not greater than $\cos^{-1}(d/a)$, where $AB=a$. [From *Cambridge Mathematical Tripos*, 1871.]

No solution of this problem has been received.

71. Proposed by B. F. FINKEL, A.M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average volume removed by boring an inch auger-hole through a cube whose edge is e, the auger to pass through two opposite faces of the cube.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School. Chester, Pa.

The average volume required is equal to the altitude multiplied by the average base. All variations of the base will be gone through by considering a cylinder through the center. The greatest base is the ellipse formed when the cylinder is tangent to two edges of the cube. Its axis is then determined by the equations $y-z\tan\beta=0$......(1), and $x-z\tan\beta=0$......(2).

β is the angle made by the plane through the origin, and the X-axis with the Y-axis. This plane is parallel to the edge $y=\frac{1}{2}e$, $z=\frac{1}{2}e$, and at a distance of $\frac{1}{2}$ inch from this edge.

$\therefore$ $\sin\beta=\sin(\frac{1}{4}\pi-\theta)$, where $\sin\theta=(\sqrt{2}/2e)$, as follows:

$$\sin\theta : 1=\tfrac{1}{2} : \tfrac{1}{2}(e\sqrt{2}).$$

The line given by

$$\left.\begin{matrix} y\cos\beta-z\sin\beta=-\frac{1}{2} \\ x\cos\beta-z\sin\beta=-\frac{1}{2}\end{matrix}\right\}......(3).$$

x, y from (1), (2), (3), when $z=\frac{1}{2}e$ are

$$x_1=\tfrac{1}{2}e\tan\beta,\ y_1=\tfrac{1}{2}e\tan\beta, \text{ and } x_2=y_2=-\tfrac{1}{2}\sec\beta+\tfrac{1}{2}e\tan\beta.$$

$$\therefore\ \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\frac{\sec\beta\sqrt{2}}{2}=\text{semi-major axis}.$$

The semi-minor axis $\frac{1}{2}$ inch for all ellipses

$$\sec\beta=\frac{1}{\cos\beta}=\frac{1}{\cos(\frac{1}{4}\pi-\theta)}=\frac{\sqrt{2}}{\cos\theta+\sin\theta}=\frac{2e}{\sqrt{(2e^2-1)}+1}.$$

$\therefore\ \sec\beta=\dfrac{e[\sqrt{(2e^2-1)}-1]}{e^2-1}$. $\quad\therefore\ \dfrac{e\sqrt{(2)}[\sqrt{(2e^2-1)}-1]}{2(e^2-1)}$ is semi-major axis.

$$\therefore\ \text{average volume is } \tfrac{1}{2}e\left(\frac{e\sqrt{(2)}[\sqrt{(2e^2-1)}-1]}{2(e^2-1)}+\tfrac{1}{2}\right)\tfrac{1}{2}\pi=\triangle.$$

$\therefore\ \triangle=\dfrac{\pi e}{8(e^2-1)}[e\sqrt{2}(\sqrt{2e^2-1}-1)+e^2-1]=5.4345$ cubic inches, when $e=5$ inches,

72. Proposed by B. F. FINKEL, A.M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A rod is broken at random into four pieces; find the chance that no one of the pieces is greater than the sum of the other three. [From *C. Smith's Treatise on Algebra*, p. 528.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let a=length of rod.

By conditions of problem no part can be greater than $\frac{1}{2}a$.

Let $ABCD-G$ be a cube side a.

Let $Abcd-g$ be a cube side $\frac{1}{2}a$.

For favorable cases the points are confined to the smaller cube.

$$\therefore\ \text{chance}=\frac{(\frac{1}{2}a)(\frac{1}{2}a)(\frac{1}{2}a)}{(a)(a)(a)}=\tfrac{1}{8}.$$

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

On an average 1 vessel out of every n is wrecked. Find the chance that out of m vessels expected p at least will arrive safely.

I. Solution by the PROPOSER.

The chance of a vessel arriving is $[(n-1)/n]$.

The chance of a vessel not arriving is $1/n$.

The event will happen if, m, $(m-1)$, $(m-2)$, $(m-3)$, $(m-4)$, ... down to p vessels arrive.

Thus the required chance is the sum of the first $(m-p+1)$ terms in the expansion of

$$\left(\frac{n-1}{n}+\frac{1}{n}\right)^m=\left(\frac{n-1}{n}\right)^m+\frac{m}{1}\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{m-1}+\frac{m(m-1)}{2!}\left(\frac{1}{n}\right)^2\left(\frac{n-1}{n}\right)^{m-2}$$
$$+\ \ldots\ldots\frac{m\,!}{p!(m-p)!}\left(\frac{1}{n}\right)^{m-p}\left(\frac{n-1}{n}\right)^p.$$

If $n=10$, $m=5$, $p=3$, we get chance$=(\frac{9}{10})^5+5(\frac{1}{10})(\frac{9}{10})^4+10(\frac{1}{10})^2(\frac{9}{10})^3$ $=\frac{12393}{12500}$.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Ia.

For convenient notation put $n-1=k$.

Then the probability that any given vessel will be wrecked is $1/n$, that it will be saved, k/n.

The probability that all the m vessels will be saved is k^m/n^m.

The probability that any particular vessel will be lost while the others are saved is $(k/n)^{m-1}(1/n)$.

There are m vessels. Hence the probability that *one* vessel will be lost while the others are saved is $m(k^{m-1}/n^m)$.

The probability that any two particular vessels will be lost while the others are saved is $(k/n)^{m-2}(1/n)^2$.

The number of different combinations for two vessels to be saved and the rest lost is $\frac{m(m-1)}{2!}$.

Hence the probability that two vessels will be lost is $\frac{m(m-1)}{2!}\frac{k^{m-2}}{n^m}$.

In like manner the probability that 3 vessels will be lost and $m-3$ saved is $\frac{m(m-1)(m-2)k^{m-3}}{n^m 3!}$ and that 1, 2, 3, ... or $m-p$ vessels will be lost is

$$P=\frac{1}{n^m}\left(k^m+mk^{m-1}+\frac{m(m-1)k^{m-2}}{2!}\ldots\ldots+\frac{m(m-1)\ldots\ldots(p-1)k^p}{(m-p)!}\right.$$

If $p=m$, $P=\left(\frac{k}{n}\right)^m$. If $p=0$, $P=\frac{(k+1)^m}{n^m}=1$, as it evidently should.

III. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

The chance that any particular ship arrives is $\frac{n-1}{n}$. The chance that any particular set of p ships arrives is $\left(\frac{n-1}{n}\right)^p$; and the chance that the other $m-p$ ships fails is $\left(\frac{1}{n}\right)^{m-p}$. Hence the chance that any particular set of p ships alone arrives is $\left(\frac{n-1}{n}\right)^p\left(\frac{1}{n}\right)^{m-p}$. But out of m ships p can be selected in $\frac{m(m-1)\ldots\ldots(m-p+1)}{1.2\ldots\ldots p}$ ways. Hence the chance that some possible set of p ships will arrive is

$$\left(\frac{n-1}{n}\right)^p\cdot\left(\frac{1}{n}\right)^{m-p}\cdot\frac{m(m-1)\ldots\ldots(m-p+1)}{1.2\ldots\ldots p}\ldots\ldots(1).$$

Writing $p+1$, $p+2$, ...m, successively, in place of p in (1) will give

the chances that $p+1$, $p+2$, ... m ships will arrive. The sum of these added to (1) gives the chance that *at least* p ships out of m will arrive.

Also solved by *ELMER SCHUYLER*.

MISCELLANEOUS.

71. Proposed by GUY B. COLLIER, 1901 Union, 27 Middle Section of South College, Schenectady, N. Y.

Find the locus of any point on the front sprocket of a bicycle during one revolution of the hind wheel (any gear may be assumed).

I. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond, Va.

Suppose the front sprocket revolves e times as fast as the hind wheel of the bicycle (if, to fix the idea, the gear be 70, then $e=\frac{2}{5}$), and take as axis of x the line on which the bicycle rolls, and as axis of y a line perpendicular to this, and passing through P when it was in its lowest position. Then let the hind wheel turn through an angle φ (the sprocket turning through an angle $e\varphi$) and the *center* of the sprocket has moved forward a distance $a\varphi$ and we obtain at once

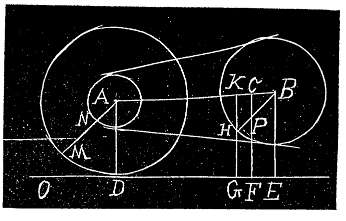
$$x=a\varphi-r\sin(e\varphi), \quad y=a-r\cos(e\varphi),$$

where r=radius of sprocket, a=radius of hind wheel, and φ=angle through which the hind wheel has turned. Expressed as a single equation, this is

$$x=(a/e)\cos^{-1}\frac{a-y}{r}-r\sqrt{1-\left(\frac{a-y}{r}\right)^2}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let O be the origin, $OF=x$, $PF=y$, P=any point in the front sprocket, g=gear, $AD=a$, radius of hind wheel, $AB=b$, $BP=c$, and also let $BE=AD=a$, $\angle MAD=\theta$.



Then $BH/AN=g/2a$.

$\therefore\ \angle HBE=(2a/g)\theta$.

$\therefore\ x=OD+DE-FE=OD+AB-BC$.

$\therefore\ x=b+a\theta-c\sin(2a\theta/g)$.

$\therefore\ y=a-c\cos(2a\theta/g)$.

$\therefore\ x=b+(\frac{1}{2}g)\cos^{-1}\left(\frac{a-y}{c}\right)-\sqrt{c^2-(a-y)^2}$ is the equation to the locus.

s=length for one revolution of hind wheel.

$$\therefore\ s=(1/g)\int_0^{2\pi}\sqrt{a^2g^2+4a^2c^2-4a^2cg\cos(2a\theta/g)}\,d\theta$$

$$=(a/g)\int_0^{2\pi}\sqrt{(g+2c)^2-8cg\cos^2(a\theta/g)}\,d\theta.$$

Let $\frac{1}{2}\pi-\phi=a\theta/g$.

$$\therefore\ s=(g+2c)\int_{(\pi/2g)(g-4a)}^{\frac{1}{2}\pi}\sqrt{1-\frac{8cg}{(g+2c)^2}\sin^2\phi}\,d\phi$$

$$=(g+2c)E^{\frac{1}{2}\pi}_{(\pi/2g)(g-4a)}\left(\frac{2\sqrt{(2cg)}}{g+2c},\ \phi\right).$$

Also solved by *WALTER H. DRANE.*

PROBLEMS FOR SOLUTION.

ARITHMETIC.

118. Solution by J. F. TRAVIS, Student in Ohio State University, Columbus, O.

The present worth of a note due January 1, 1896, was \$74,200, when discounted at 4% true discount. The present worth of another note, due July 1, 1896, whose face value was the same as that of the first note, was \$68900 when discounted at 8% true discount. Find the face of the notes and the date when given, supposing the second note to have been given the same day the first note was. Solve by arithmetic.

119. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$1.\frac{9}{.001}$. The selling price \$1,000. What is the gain per cent.?

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

ALGEBRA.

107. Proposed by CHARLES MYERS, Canton, Ohio.

Given $xyz=18\ldots.(1)$; $x^2+y^2+z^2=33\ldots.(2)$, and $(x^2-yz)^3+(y^2-xz)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)(z^2-xy)=6561\ldots.(3)$, to find x, y, and z.

108. Proposed by GEORGE LILLEY, Ph.D., LL.D., Professor of Mathematics, State University, Eugene, Or.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent., annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the endorsement on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$?

*** Solutions of these problems should be sent to J. M. Colaw not later than November 10.

GEOMETRY.

126. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Through any fixed point O draw two straight lines at right angles. Let one line cut a given circle at Q, the other at R. Find, by Euclidean methods, the locus of the foot of the perpendicular from O upon the chord QR. Give complete analysis and discussion. Solve also by coördinate geometry.

127. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

The equation to the plane through the extremities, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), of conjugate diameters of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=1.$$

⁂ Solutions of these problems should be sent to B. F. Finkel not later than November 10.

CALCULUS.

97. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

An augur hole radius r is bored through a prolate spheroid; the axis of the augur passing through the center, perpendicular to the major axis. Find the volume removed.

98. Proposed by CHARLES CARROLL CROSS, Whaleyville, Va.

On the circumference of a fixed circle radius R rolls a circle radius r. Required the length of the curve described by a point on the circumference of the rolling circle; (1) when the circle rolls on the inside; (2) when the circle rolls on the outside of the circumference of the fixed circle.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than November 10.

MECHANICS.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The side AB of the parallelogram $ABCD$ will be a principal axis at the point which divides the distance between the middle point and the foot of the perpendicular from the middle-point of the opposite side in the ratio 2 : 1. The principal moments of inertia about this point are $\frac{1}{3}mb^2\sin^2\beta$, $\frac{1}{36}m(3a^2+4b^2\cos^2\beta)$, where $\beta=\angle A$.

98. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A spool, with light thread wound around, is placed upon a rough table so that the thread will emerge from beneath the spool. The thread is passed over a smooth pulley at end of table and a weight attached, the pulley being so adjusted that thread is parallel to surface of table. If friction between spool and table is sufficient to prevent slipping, de-

termine motion of spool and weight. [From problems in Mechanics at Harvard University.]

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

AVERAGE AND PROBABILITY.

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A box contains 100 balls marked from 1 to 100. 13 balls are drawn at random. What is the chance that the balls marked from 1 to 10 are included in the 13 drawn ?

81. Proposed by LON C. WALKER, Graduate of Leland Stanford, Jr., University, Palo Alto, Cal.

Find (1) the mean distance of all points on a side of an equilateral triangle from the opposite vertex; and (2), the average length of a line drawn at random across an equilateral triangle.

*** Solutions of these problems should be sent to B. F. Finkel not later than November 10.

MISCELLANEOUS.

81. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A cask in the form of a middle frustum of a spheroid, middle diameter $2b$, end diameters each $2c$, length $2d$, is lying in a horizontal position. The distance from middle of top to water is $b+e$, $e<b-c$. How much water is in the cask ?

82. Proposed by A. H. BELL, Hillsboro, Ill.

Four spheres of equal radii$=r=5$, are in contact, and form a triangular period. How large is the sphere that can be placed in middle and be in contact with the four spheres ?

*** Solutions of these problems should be sent to J. M. Colaw not later than November 10.

EDITORIALS.

Prof. J. O. Mahoney has been elected teacher of mathematics in the Dallas High School, Dallas, Texas.

Mr. Peter Field, Fellow in Cornell University, has been appointed Professor of Mathematics in Carthage College, Carthage, Ill.

Dr. J. V. Westfall, Honorary Fellow in Cornell University, has been appointed Senior Instructor in Mathematics in the Iowa State University.

Miss Mary M. Blaine, B. S. (Drury College), has been given a scholarship in the University of Pennsylvania, and has gone there to pursue a course of study in mathematics.

The Franklin Institute of Philadelphia has lately created the office of mathematical contributor to the Physical and Astronomical Section, and has appointed Dr. G. B. M. Zerr as the contributor for the coming year.

This issue of this MONTHLY has been unavoidably delayed. Subsequent issues will be out on time. The next issue will contain more than the usual number of pages, as an article of unusual interest, by Dr. Halsted, is to appear.

Dr. L. E. Dickson, formerly Assistant Professor of Mathematics in the University of California, has been elected Associate Professor of Mathematics in the University of Texas. Dr. Halsted is to be congratulated in being able to call to his assistance one of the ablest young mathematicians in the country. With Dr. Halsted noted for his valuable work in non-Euclidean Geometry, and Dr. Dickson a recognized authority on Group Theory, the University of Texas will be able to offer as good courses in mathematics as are offered anywhere in this country.

The Sixth Summer Meeting of the American Mathematical Society met at Columbus, Ohio, August 25th and 26th. The meeting was well attended, and the papers read and discussed were of great value. Dr. J. V. Collins read a paper on "A relation between point and vector analysis"; Dr. E. H. Moore presented Prof. Frank Morley's paper, "On the generalization of Desargues' theorem," and one of his own "On certain crinkly curves"; Dr. G. A. Miller read a paper "On groups that are the direct products of two subgroups"; and Dr. L. E. Dickson read two papers, one on "A new definition of the general Abelian linear group," the other on "Definitions of various linear groups as groups of isomorphisms." Dr. Halsted's able report on Non-Euclidean Geometry was read before Section A of the Association for the Advancement of Science on Tuesday, and was greatly appreciated by his audience. Dr. Alexander Macfarlane, president of Section A, read a very interesting paper on "The fundamental principles of algebra." Dr. Macfarlane gave a very interesting and exhaustive treatment of this subject, tracing the important advances in the philosophy of the fundamental principles of algebra which have been made in the present century.

At this meeting we had the pleasure of meeting a number of our good friends. Among those, whom we had long known through correspondence, but now have the pleasure of knowing personally, were Dr. Halsted, of the University of Texas ; Prof. Ormond Stone, of the University of Virginia ; Prof. R. S. Woodward, of Columbia ; Prof. D. V. Bohannon, of the Ohio State University ; Dr. J. V. Collins, of the State Normal School of Wisconsin ; Prof. W. W. Beman, of the University of Michigan ; Mr. J. W. Young, Graduate Student, Ohio State University ; Dr. Alexander Macfarlane, Lehigh University ; and Prof. F. E. Miller, of Otterbein University.

BOOKS AND PERIODICALS.

Stereoscopic Views of Solid Geometry, with Reference to Wells' Essentials of Solid Geometry. Set, 96 views. Price, 60 cents per set. Boston: D. C. Heath & Co.

These views are very beautiful and attractive. They are printed on heavy card board, having a black background and white lines. B. F. F.

A Treatise on the Theory and Solutions of Algebraical Equations. By John Macnie, M. A., Professor of the Latin Language and Literature, University of North Dakota. Large 8vo. Half Leather Back. 184 pages. Price, $1.20, net. Chicago: A. S. Barnes & Co.

This work discusses, in a very careful and accurate way, Cauchy's Theorem, Sturm's Theorem, Horner's Method, Fourier's Theorem, Symmetric Functions of the Roots, Elimination, etc., etc. B. F. F.

New Higher Algebra. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Half Leather Back. 446 pages. Price, $1.32. Boston: D. C. Heath & Co.

The first 358 pages of this book are identical with the corresponding pages of the author's *Essentials of Algebra.* The new matter added is Compound Interest and Annuities, Permutations and Combinations, Continued Fractions, Summation of Series, Theory of Equations, and Solution of Higher Equations. B. F. F.

An Elementary Treatise on the Theory of Equations. By Samuel Marx Barton, Ph. D., Professor of Mathematics, University of the South. 8vo. Cloth, 200 pages. Price, $1.50. Boston: D. C. Heath & Co.

In this book is set forth the elements of Determinants and the Theory of Equations in quantity and in form suitable for use in undergraduate work in our colleges and universities. Part I treats of Determinants, and Part II of the Theory of Equations. The book is well adapted to the purpose for which it was written. B. F. F.

Memoir on the Substitution Groups whose Degree does not Exceed Eight. By Dr. G. A. Miller, of Cornell University. Reprinted from *American Journal of Mathematics*, Vol. XXI, No. 4, pages 287—338.

In this memoir Dr. Miller has discussed the Substitution Groups whose degree does not exceed eight. Dr. Miller, though a young man, is one of the leading authorities on Group Theory in this country. His contributions on this subject are characterized by accuracy and simplicity of treatment. This memoir is one of great value to the student of Groups, and will become a permanent part of the literature on the subject. B. F. F.

A Report on Greene County. Part I. of Geological Survey of Missouri, by Edward M. Shepard, M. A., Professor of Geology in Drury College. 8vo. Cloth. 246+iv pages. Printed by the State of Missouri.

Among the interesting points about this book may be mentioned the fact that it is wholly a pioneer work, and that the author has discovered and named a number of geological formations hitherto unknown. Among these are the "Republic chert" and the "Graydon sandstone" and its connection with an ancient river system, notably the fossil river bed which he has named the "Schoolcraft river," extending through the whole western half of the district and northward to the Missouri river, into which its ancient waters

poured. He was the first to discover the presence of Devonian rocks in Southwest Missouri, from which he has named three horizons new to science, viz., the Phelps sandstone, the Sac limestone, and the King limestone. He was also the first to demonstrate the presence of the great flexuring and faulting in the Ozark uplift. The discussion of the ore deposits of the district is particularly valuable, and the large list of locations where lead and zinc have been found in the past will be of great interest to prospectors in this region.

Professor Shepard's extensive travels in all parts of the United States, in Cuba, in the Hawaiian Islands, in Australia, in New Zealand, and many other places, has enabled him to compare the region of which he has written with those of the highest geological interest. The book has many fine illustrations and a number of excellent geological maps.

B. F. F.

The Fundamental Principles of Algebra. An Address by Dr. Alexander Macfarlane, Vice President and Chairman of Section A of the Association for the Advancement of Science, delivered at Columbus, Ohio, August 21, before Section A of the Association. Pamphlet, 32 pages.

In this address Dr. Macfarlane reviews historically and critically the advances that have been made respecting the fundamental principles of Algebra. The work done along this line by Servais, Babbage, Herschel, Peacock, and others, is brought out. Mention is made of the elaborate treatise on Algebra by G. Chrystal, and several pages are devoted to a review of Mr. Whitehead's Treatise on Universal Algebra. The address should be read by everyone of our readers not having had the privilege of hearing it. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year, in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

A magazine of the highest literary and artistic merit. Read Mark Twain's article on "Christian Science and the Book of Mrs. Eddy" in the October number, and enjoy the pleasure of a good hearty laugh. B. F. F.

The Open Court. A Monthly Magazine Devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea. Dr. Paul Carus, Editor. T. J. McCormack, Assistant Editor. Price, $1.00 per year. Chicago: The Open Court Publishing Co.

In the October number are a number of interesting articles relating to Germany.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. New York: The Review of Reviews Co.

The American Monthly Review of Reviews for September contains a remarkably attractive group of contributed articles. The timeliness of the subjects treated is seen by a glance at the table of contents. The war in the Philippines is summed up by John Barrett; the outcome of The Hague conference is set forth by W. T. Stead; the subject of trusts is discussed by George E. Roberts and by Henry Macfarland; Hezekiah Butterworth writes of "The Future Value of the New England Farm," while Prof. L. H. Bailey answers affirmatively the question, "Does Farming Pay?" Sylvester Baxter tells of the progress made by the State of Massachusetts in her public library system, and Gilbert K. Harroun describes the work of the Cuban Educational Association of the United States; a sketch of "The New Secretary of War" is contributed by Henry Macfarland, while Dr. William Hayes Ward writes of Colonel Ingersoll, and Erica Glenton of the late Grand Duke George of Russia. B. F. F.

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REPORT ON PROGRESS IN NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

[Read at Columbus Meeting of the American Association for the Advancement of Science, August 21—26.]

It marks an epoch in the history of mathematics that at a meeting of a great Association for the advancement of science there should be presented by invitation a Report on non-Euclidean geometry.

Its two creators, Lobachévski who mis-named it Imaginary Geometry, and Bolyai János, under the nobler name Science Absolute of Space, failed utterly while they lived, to win any appreciative attention for what is to-day justly honored as one of the profoundest advances of all time. The only recognition, the only praise of the achievement of Lobachévski ever printed in his lifetime was by Bolyai Farkas the father of his brilliant young rival, and appeared in a little book with no author's name on the title page, and which we have no evidence that Lobachévski ever saw, a little book so rare that my copy is probably the only one on the western continent.

When after more than forty years their names were rescued from oblivion by Baltzer and Hoüel in 1866, still envious Time gave them back only with an aspersion against the genuineness of their originality. A cruel legend tarnished still their fame so long delayed, so splendidly deserved.

Even when their creation had reached the high dignity of made the subject of courses of lectures for consecutive semesters at the University of Göttingen, yet on page 175 of the second impression of these lectures, 1893, we still find Felix Klein saying, "Kein Zweifel bestehen kann, dass Lobatscheffsky

sowohl wie Bolyai die Fragestellung ihrer Untersuchungen der Gaussischen Anregung verdanken."

It is a privilege to begin my Report by announcing the rigorous demonstration that this ungenerous legend is untrue. This point need not further delay us, since it has been treated by me at length in *Science*, N. S. Vol. IX., No. 232, pages 813—817, June 9, 1899.

What a contrast to the pathetic neglect of its creators, Lobachévski dying blind, unrecognized, without a single follower, Bolyai János dying of disgust with himself and the world, lies in the fact that less than a year ago our American magazine the *Monist* secured from the famous Poincaré at great cost a brilliant contribution to this now universally interesting subject, which I had the honor, through my friend T. J. McCormack, of reading in the original French manuscript.

This extraordinary paper, published only in English translation, appears in the *Monist*, Vol. 9, No. 1, October, 1898, pages 1—43. In the first section of his greatest work Lobachévski says: "*Juxtaposition* (contact) is the distinctive characteristic of solids, and they owe to it the name *geometric solids*, when we retain this attribute, taking into consideration no others whether essential or accidental.

Besides bodies, for example, also time, force, velocity are the object of our judgment; but the idea contained in the word juxtaposition does not apply thereto. In our mind we attribute it only to solids, in speaking of their composition or dissection into parts.

This simple idea, which we have received directly in nature through the senses, comes from no other and consequently is subject to no further explanation. Two solids A and B, touching one another, form a single geometric solid C, in which each of the component parts A, B appears separate without being lost in the whole C. Inversely, every solid C is divided into two parts A and B by any *section* S.

By the word section we understand here no new attribute of the solid, but again a juxtaposition, expressing thus the partition of the solid into two juxtaposed parts.

In this way we can represent to ourselves all solids in nature as parts of a single whole solid which we call space."

Poincaré starts off somewhat differently. He says: "We at once perceive that our sensations vary, that our impressions are subject to change. The laws of these variations were the cause of our creating geometry and the notion of geometrical space.

Among the changes which our impressions undergo, we distinguish two classes:

(1) The first are independent of our will and not accompanied by muscular sensations. These are *external changes* so called.

(2) The others are voluntary and accompanied by muscular sensations. We may call these *internal changes*.

We observe next that in certain cases when an external change has modi-

fied our impressions, we can, by voluntarily provoking an internal change, re-establish our primitive impressions. The external change, accordingly, can be *corrected* by an internal change. External changes may consequently be subdivided into the two following classes :

1. Changes which are susceptible of being corrected by an internal change. These are *displacements.*

2. Changes which are not so susceptible. These are *alterations.* An immovable being would be incapable of making this distinction. *Such a being, therefore, could never create geometry,*—even if his sensations were variable, and even if the objects surrounding him were movable.''

How like what Lobachévski said more than sixty years before: ''We cognize directly in nature only motion, without which the impressions our senses receive are not possible. Consequently, all remaining ideas, for example, geometric, are created artificially by our mind, since they are taken from the properties of motion ; and therefore space in itself, for itself alone, does not exist for us.''

Poincaré continues: ''*The aggregate of displacements is a group.*'' At once rise before us the great names Riemann, Helmholtz, Sophus Lie. In fact Poincaré's next section is merely a restatement of part of Riemann's marvellous address, published 1867, on the hypothesis at the basis of geometry.

Again, though the work of Helmholtz did not contain the group idea, yet it had put the problem of non-Euclidean geometry into the very form for the instrument of Sophus Lie, who calls it the Riemann-Hemholtz Space-problem.

To the genius of Helmholtz is due the conception of studying the essential characteristics of a space by a consideration of the movements possible therein.

Felix Klein it was who first called the attention of Lie to this work of Helmholtz, before then unknown to Lie, and pointed out its connection with Lie's Theory of Transformation Groups, inciting him to a group-theory investigation of the problem. In 1886 Lie gave briefly his weightiest results in a note: ''Bemerkungen zu v. Helmholtz' Arbeit ueber die Thatsachen, die der Geometrie zu Grunde liegen,'' in the Berichte of the Saxon Academy, where in 1890 he gave his completed work in two papers Ueber die Grundlagen der Geometrie (pages 284—321, 355—418). The whole investigation published in Volume III of his ''Theorie der Transformationsgruppen,'' 1893, was in 1897 awarded the first Lobachévski Prize. Felix Klein declared that it excels all comparable works so absolutely that a doubt about the award could scarcely be possible. Lie gives two solutions of the problem. In the first he investigates in space a group possessing free mobility in the infinitesimal, in the sense, that if a point and any line-element through it be fixed, continuous motion shall still be possible ; but if besides any surface-element through the point and line-element be fixed then shall no continuous motion be possible. The groups in tri-dimensional space possessing in a real point of general position this free mobility Lie finds to be precisely those characteristic of the Euclidean and the two non-Euclidean geometries. Strangely enough, for the seemingly analogous and simpler case of the plane or two-dimen-

sional space these are not the only groups. There are others where the paths of the infinitesimal transformations are spirals. Without the group idea, Helmholtz had reached this reality, and as a consequence concluded that also to characterize our tri-dimensional spaces a new condition, a new axiom, was needed, that of *monodromy*. It is one of the most brilliant results of Lie's second solution of the space problem, that starting from transformation-equations with three of Helmholtz's four assumptions he proves that the fourth, the famous "Monodromie des Raumes," is, in space of three dimensions, wholly superfluous. What a demonstration of the tremendous power of Lie's Group Theory! Lie's method in general, as it appears in the Berichte, is the following:

Consider a tri-dimensional space, in which a point is defined by three quantities, x, y, z.

A movement is defined by three equations: $x_1 = f(x, y, z)$; $y_1 = \varphi(x, y, z)$; $z_1 = \psi(x, y, z)$.

By this transformation an assemblage, A, of points (x, y, z) becomes an assemblage, A', of points (x_1, y_1, z_1).

This represents a movement which changes A to A'. Now make, in regard to the space to be studied, the following assumptions:

(B) In reference to any pair of points which are moved there is *something* which is left unchanged by the motion. That is, after an assemblage of points, A, has been turned by a single motion into an assemblage of points, A', there is a certain function, Ω, of the coördinates of any pair of the old points (x_1, y_1, z_1), (x_2, y_2, z_2) which equals that same function, Ω, of the corresponding new coördinates (x'_1, y'_1, z'_1), (x'_2, y'_2, z'_2); that is, $\Omega(x'_1, y'_1, z'_1, x'_2, y'_2, z'_2) = \Omega(x_1, y_1, z_1, x_2, y_2, z_2)$. This *something* corresponds to the generalized idea of distance interpreted as independent of measurement by superposition of an unchanging sect as unit for length. Moreover assume

(C) If one point of the assemplage is fixed, every other point of this assemblage, *without any exception*, describes a surface (a two-dimensional aggregate). When two points are fixed, a point in general (exceptions being possible) describes a curve (a one-dimensional aggregate). Finally if three points are fixed, all are fixed (exceptions being possible). Then Lie proves exhaustively that the group consists either of all motions of Euclidean space or of all motions of non-Euclidean space.

The result is a remarkable one, demonstrating that the group of Euclidean motions and the group of non-Euclidean motions are, in tri-dimensional space, the only groups in which exists in the strict sense of the word free mobility. Thus free motion in the strict meaning of the word can happen in three and only three spaces, namely, the traditional or Euclidean space, and the spaces in which the group of movements possible is the projective group transforming into itself one or the other of the surfaces of the second degree $x^2 + y^2 + z^2 \pm 1 = 0$.

To the fundamental assumption which completely characterizes these three groups Lie gives also this form:

"If any real point y_1^0, y_2^0, y_3^0 of general position is fixed, then all real

points x_1, x_2, x_3, into which may still shift another real point x_1^0, x_2^0, x_3^0, satisfy a real equation of the form: $W(y_1^0, y_2^0, y_3^0 ; x_1^0, x_2^0, x_3^0 ; x_1, x_2, x_3)=0$, which is not fulfilled for $x_1=y_1^0$, $x_2=y_2^0$, $x_3=y_3^0$, and which represents a real surface passing through the point x_1^0, x_2^0, x_3^0.

About the point y_1^0, y_2^0, y_3^0 may be so demarcated a triply extended region, that on fixing the point y_1^0, y_2^0, y_3^0, every other real point x_1^0, x_2^0, x_3^0 of the region can yet shift continuously into every other real point of the region, which satisfies the equation $W=0$ and which is joined to the point x_1^0, x_2^0, x_3^0 by an irreducible continuous series of points."

It is a satisfaction to the world of science that Lie's vast achievements were recognized while he lived. Poincaré accepts and expounds his doctrine, saying in the article already mentioned: "The axioms are not analytical judgments *a priori*; they are conventions. . . . Thus our experiences would be equally compatible with the geometry of Euclid and with a geometry of Lobachévski which supposed the curvature of space to be very small. We choose the geometry of Euclid because it is the simplest.

If our experiences should be considerably different, the geometry of Euclid would no longer suffice to represent them conveniently, and we should choose a different geometry."

When on November 3, 1897, the great Lobachévski prize was awarded to Lie, three other works were given honorable mention. The first of these is a thesis on non-Euclidean geometry by M. L. Gérard of Lyons. Lovers of the non-Euclidean geometry are naturally purists in geometry, and keenly appreciate Euclid's using solely such figures as he has rigorously constructed. The understand that problems of construction play an essential part in a scientific system of geometry. Far from being solely, as our popular text-books suppose, practical operations, available for the training of learners, they have in reality, as Helmholtz declares, the force of existential propositions. Therefore is evident the high import of Gérard's work to establish the fundamental propositions of non-Euclidean geometry without hypothetical constructions other than the two assumed by Euclid: 1. Through any two points a straight line can be drawn. 2. A circle may be described from any given point as a center with any given sect as radius. Gérard adds explicitly the two assumptions: 3. A straight line which intersects the perimeter of a polygon in a point other than one of its vertices intersects it again ; 4. Two straights, or two circles, or a straight and a circle, intersect if there are points of one on both sides of the other.

Upon these four hypotheses, perfecting a brilliant idea of Battaglini (1867), Gérard establishes the relations between the elements of a triangle.

Lobachévski never explicitly treats the old problems changed by transference into the new geometric world, such as "Through a given point to draw a parallel to a given straight"; nor yet the seemingly impossible problems now in it capable of geometric solution, such as "To draw to one side of an acute angle the perpendicular parallel to the other side"; "To square the circle."

These would be sought in vain in the two quarto volumes of Lobachevski's

collected works. Bolyai János in his all too brief two dozen pages gives solutions of them startling in their elegance.

But in establishing his theory, he uses, for the sake of conciseness, the principle of continuity even more freely than does Lobachévski.

Gérard, in the second part of his memoir, gives the elements of non-Euclidean analytic geometry, and in the third part a strict treatment of equivalence.

Even Euclid in proving his I. 35, "Parallelograms on the same base, and between the same parallels, are equal to one another," does not show that the parallelograms can be divided into pairs of pieces admitting of superposition and coincidence. He uses rather the assumption explicitly set forth by Lobachévski, "Two surfaces are equal when they arise from the mating or separating of equal parts." But Creswell in his *Treatise of Geometry*, showed how to cut the parallelograms into parts congruent in pairs. The same can be done for Euclid I. 43, "The complements of the parallelograms which are about the diagonal of any parallelogram are equal." Hence we may use the definition: Magnitudes are equivalent which can be cut into parts congruent in pairs. This method I applied to the ordinary Euclidean geometry in my Elementary Synthetic Geometry before the appearance of Gérard's work, where it is extended to the non-Euclidean.

Regarding the first assumed construction of Euclid and Gérard: "A straight line can be drawn through any two points," W. Burnside has given us a charming little paper in the Proceedings of the London Mathematical Society, Vol. XXIX, pages 125—132 (December 9, 1897) entitled "The Construction of the Straight Line joining Two Given Points." Euclid's postulate implies the use of a ruler or straight-edge of any required finite length. The postulate is clearly not intended to apply to the case in which the distance between the two points is infinite. In fact Euclid I. 31 gives a compass and ruler construction for the line when one of the points can be reached while the other cannot. The other exceptional case when neither point can be reached, *i. e.*, when two given points are the points at infinity on two non-parallel lines, is not dealt with by Euclid.

In elliptic space any one point can be reached from any other by a finite number of finite operations. The line joining two given points can therefore be always constructed with the ruler alone. In hyperbolic space, if we deal with projective geometry, we must assume that *every* two straight lines in a plane determine a point. When the two straight lines are non-intersectors the point can neither be a finite point nor a point at infinity. Such a point is termed an "ideal" point. The problem of constructing the straight line joining two given points involves therefore three further cases, namely, (IV) that in which one of the points is a finite point and the other an ideal point; (V) that in which one is a point at infinity and the other an ideal point; (VI) that in which both points are ideal points.

It is a pleasure to signal the appearance, within the past year, of the second

volume of the exceedingly valuable work of Dr. Wilhelm Killing, "Einfuehrung in die Grundlagen der Geometrie," (Paderborn, 1898).

With Killing's name will be associated the tremendous difference living geometers find between the properties of a finite region of space, and the laws which pertain to space as a whole. Of the word *direction* he says, it can only be given a meaning when the whole theory of parallels is already presupposed.

The pseudo-proof of the parallel postulate still given in current text-books, for example by G. C. Edwards in 1895, Killing calls the Thibaut proof, saying that it has especial interest because its originator, who was professor of mathematics at Göttingen with Gauss, published the attempt at a time, 1818, when Gauss had already called attention to the failure of attempts to prove this postulate, and declared that we had not progressed beyond where Euclid was two thousand years before.

But Killing is here in error when he supposes Thibaut the originator of this popular pseudo-proof. It was given in 1813 by Playfair in his edition of Euclid, in a Note to I. 29. It was very elegantly shown to be a fallacy by Colonel T. Perronet Thompson, of Queen's College, Cambridge, in a remarkable book called "Geometry without Axioms," of which the third edition is dated 1830, a book seemingly unknown in Germany, since Engel and Staeckel copy from Riccardi the title (with the mistake "first books" for "first book") under the date 1833, which is the date of the fourth edition.

Killing has won an important place by investigating the question, what varieties of connection of space are compatible with the different elemental arcs of constant curvature. Riemann, Helmholtz, and Lie consider only a region of space, and give analytic expressions for the vicinity of a point. If this region be extended, the question is, what kind of connection of space can result.

Killing shows there are different possibilities, really a series of topologically different forms of space with Euclidean, Lobachévskian, Riemannian geometry in the bounded, simply connected region.

The germinal idea is due to Clifford, who, in an unprinted address before the Bradford meeting of the British Association (1873), "On a surface of zero curvature and finite extent," and also by a remark in his paper "Preliminary sketch of biquaternions," called attention to a recurrent surface in single elliptic space, which has everywhere zero for measure of curvature, yet is nevertheless of finite area.

Similarly complete universal spaces are found of zero or negative measure of curvature, which nevertheless are only of finite extent. Since there is no way of proving that the whole of our actual space can be moved in itself in ∞^6 ways, it may possibly be, after all, one of these new Clifford spaces. Free mobility of bodies may only exist while they do not surpass a certain size. Killing devotes an interesting section, over seven pages, to Legendre's definition of the straight line as the shortest distance between two points. He emphasizes three principal reasons why this is inadmissible. These are, (a) since the possibility of measurement for all lines is presumed beforehand, which is not allowable ; (b) since

before the execution of the measurement there must be a measuring standard, but this is first given by the straight line ; (c) since the existence of a minimum is not evident, on the contrary can be demanded only as an assumption.

The first objection was always conclusive, yet it strengthens every day, for our new mathematics knows of lines, real boundaries between two parts of a plane, to which the idea of length is inapplicable.

Under the title "Universal Algebra" one would scarcely look for a treatise on non-Euclidean geometry. Yet the first volume of Whitehead's admirable work (Cambridge, 1898, pages 586) devotes more than 150 pages to an application of Grassmann's Calculus of Extension to hyperbolic, elliptic, parabolic spaces. So devoted is he, that we find him saying : "Any generalization of our space conceptions, which does not at the same time generalize them into the more perfect forms of Hyperbolic or Elliptic Geometry, is of comparatively slight interest." He emphasizes the fact that the three-dimensional space of ordinary experience can never be proved parabolic. "The experience of our senses, which can never attain to measurements of absolute accuracy, although competent to determine that the space-constant of the space of ordinary experience is greater than some large value, yet cannot, from the nature of the case, prove that this space is absolutely Euclidean."

From the many important contributions by Whitehead may be singled out as especially timely his development of a theorem by Bolyai János to which F. S. Macauly called special attention in the second of his able articles entitled, John Bolyai's "Science Absolute of Space" (*The Mathematical Gazette*, No. 8, July, 1896, pages 25—31 ; No. 9, October, 1896, pages 49—60). Macauly says, page 53, "Finally follows a theorem (§21), which is undoubtedly the most remarkable property of hyperbolic space, that the sum of the angles of any triangle formed by L-lines on an F-surface is equal to two right angles. On this theorem Bolyai remarks (Halsted's Bolyai, 4th Ed. page 18) : 'From this it is evident that Euclid's Axiom XI, and all things which are claimed in geometry and plane trigonometry hold good *absolutely* in F, L-lines being substituted in place of straights. Therefore the trigonometric functions are taken here in the same sense (are defined here to have the same values) as in Σ (as in Euclidean geometry) ; and the periphery of the circle, of which the L-form radius$=r$ in F, is$=2\pi r$, and likewise the area of circle with radius r (in F)$=\pi r^2$ (by π understanding half the periphery of circle with radius 1 in F, or the known 3.1415926....)'."

Whitehead in his Universal Algebra, §262, recurs to this important point, saying, "The idea of a space of one type as a locus in space of another type, and of dimensions higher by one, is partly due to J. Bolyai, and partly to Beltrami. Bolyai points out that the relations between lines formed by great circles on a two-dimensional limit-surface are the same as those of straight lines in a Euclidean plane of two dimensions. Beltrami proves by the use of the pseudosphere, a hyperbolic space of any number of dimensions can be considered as a locus in Euclidean space of higher dimensions. There is an error, popular even among

mathematicians misled by a useful technical phraseology, that Euclidean space is in a special sense flat, and that this flatness is exemplified by the possibility of a Euclidean space containing surfaces with the properties of hyperbolic and elliptic spaces. But the text shows that this relation of hyperbolic to Euclidean space can be inverted. Thus no theory of the flatness of Euclidean space can be founded upon it." Whitehead has since followed up his point in a very important and powerful paper in the Proceedings of the London Mathematical Society, Vol. XXIX, pages 275—324, March 10, 1898, entitled "The Geodesic Geometry of Surfaces in non-Euclidean Space." He there says, "The relations between the properties of geodesics on surfaces and non-Euclidean geometry, as far as they have hitherto been investigated, to my knowledge, are as follows :

It has been proved by Beltrami that the 'geodesic geometry' of surfaces of constant curvature in *Euclidean* space is the same as the geometry of straight lines in planes in elliptic or in hyperbolic space, according as the curvature of the surface is positive or negative.

The geometry of great circles on a sphere of radius ρ in elliptic space of 'space-constant' γ is the same as the geometry of straight lines in planes in elliptic space of space constant $\gamma \sin \frac{\rho}{\gamma}$.

The geometry of great circles on a sphere of radius ρ in hyperbolic space of 'space-constant' γ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sinh \frac{\rho}{\gamma}$.

The geometry of geodesics (that is, lines of equal distance), on a surface of equal distance, σ, from a plane in hyperbolic space of space-constant γ, is the same as that of straight lines in planes in hyperbolic space of space constant $\cosh \frac{\sigma}{\gamma}$.

Finally, the geometry of geodesics (that is, limit-lines), on a limit surface in hyperbolic space—which may be conceived either as a sphere of infinite radius or as a surface of equal, but infinite, distance from a plane—is the same as that of straight lines in planes in Euclidean space.

The preceding propositions are due directly, or almost directly, to John Bolyai, though, of course, he only directly treats of hyperbolic space.

From the popularization of Beltrami's results by Helmholtz, and from the unfortunate adoption of the name "radius of space curvature" for γ (here called the space constant), many philosophers, and, it may be suspected from their language, many mathematicians, have been misled into the belief that some peculiar property of flatness is to be ascribed to Euclidean space, in that planes of other sorts of space can be represented as surfaces in it. This idea is sufficiently refuted, at least as regards hyperbolic space, by Bolyai's theorem respecting the geodesic geometry of limit surfaces. For a Euclidean plane can thereby be represented by a surface in hyperbolic space.

It is the object of this paper to extend and complete Bolyai's theorem by

investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane.

Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution.

In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution. The surfaces fall into two main types; the limit surfaces form a transition case between these types. In elliptic space there is only one type of such a surface of revolution.

The same principles would enable the problem to be solved of the discovery in any kind of space of surfaces with their 'geodesic' geometry identical with that of planes in any other kind of space."

So that which Macauly designated as "undoubtedly the most remarkable property of hyperbolic space" has been by Whitehead not only generalized for hyperbolic space but extended to elliptic space.

Bolyai János seemed fully to realize the weight, the scope, the possibilities, the meaning of his discovery. He returns to it in §37, where he uses the proportionality of similar triangles in F to solve an essential problem in S (hyperbolic space). Then he adds: "Hence easily appears (L-lines being given by their *extremities alone*) also fourth and mean *terms* of a proportion can be found, and all geometric constructions which are made in Σ in plane, in this made can be accomplished in F *apart from Axiom XI*." The italics are Bolyai's, yet I find that they have not been reproduced in my published translation (the only one in English), nor in Frischauf's German, nor in Hoüel's French, nor in Fr. Schmidt's Latin text, nor in Suták's Magyar. Whitehead's researches will remind us all how great a thing it was to have reached the whole Euclidean system entirely apart from any parallel-postulate. It is a pleasure to be able to state that this was also done by Lobachévski. It is explicitly given in his first published work "O nachalah geometri" (1829). "Noviya nachala geometri" (1835) devotes to it Chapter VIII. It is also at this point, so striking as pure mathematics, that general philosophy finds itself involved. Killing, Klein, and in general the German writers, distinctly draw back from any philosophical implications. The whole matter, however, has been ably opened in "An Essay on the Foundations of Geometry," by Hon. Bertrand A. W. Russell, Fellow of Trinity College, Cambridge (1897), who has had the good fortune to be the very first to set forth the philosophical importance of von Staudt's pure projective geometry, which in its foundation and dealing with the qualitative properties of space involves no reference to quantity. I discussed this point more than twenty yeors ago in the *Popular Science Monthly*, *a propos* of Spencer's classification of the Abstract Sciences.

In a note to the first edition of his classification of the sciences (omitted in the second edition) Spencer says, "I was ignorant of this as a separate division of mathematics, until it was described to me by Mr. Hirst. It was only

when seeking to affiliate and define 'Descriptive Geometry' that I reached the conclusion that there is a negatively-quantitative mathematics as well as a positively-quantitative mathematics." As explanatory of what he wishes to mean by negatively-quantitative we quote from his Table I.: "Laws of Relations, that are Quantitative (Mathematics), Negatively: the terms of the relations being definitely-related sets of positions in space, and the facts predicated being the absence of certain quantities ('Geometry of Position')." He also says: "In explanation of the term 'negatively-quantitative,' it will be sufficient to instance the proposition that certain three lines will meet in a point, as a negatively-quantitative proposition, since it asserts the absence of any quantity of space between their intersections. Similarly, the assertion that certain three points would always fall in a straight line is 'negatively quantitative,' since the conception of a straight line implies the negation of any lateral quantity or deviation." But Sylvester has said of this very proposition that it "refers solely to position, and neither invokes nor involves the idea of quantity or magnitude."

"Projective Geometry proper," says Russell, "does not employ the conception of magnitude."

Now it is in metrical properties alone that non-Euclidean and Euclidean spaces differ. The distinction between Euclidean and non-Euclidean geometries, so important in metrical investigations, disappears in projective geometry proper. Therefore projective geometry deals with a wider conception, a conception which includes both and neglects the attributes in which they differ. This conception Mr. Russell calls "a form of externality." It follows that the assumptions of projective geometry must be the simplest expression of the indispensable requisites of all geometrical reasoning.

Any two points uniquely determine a line, *the straight*. But any two points and their straight are, in pure projective geometry, utterly indistinguishable from any other point pair and their straight. It is of the essence of *metric* geometry that two points shall completely determine a spatial *quantity*, *the sect* (German, strecke). If Mr. Russell had used for this fundamental spatial magnitude this name, or any name but 'distance,' his exposition would have gained wonderfully in clearness. It is a misfortune to use the already overworked and often misused word 'distance' as a confounding and confusing designation for a sect itself and also the measures of that sect, whether by superposition, ordinary ratio, indeterminate as depending on the choice of a unit; or by projective metrics, indeterminate as depending on the fixing of the two points to be taken as constant in the varying cross ratios.

That Mr. Russell's chapter, "A short history of metageometry," contains all the stock errors in particularly irritating form, and some others peculiarly grotesque, I have pointed out *in extenso* in *Science*, Vol. VI., pages 487—491. Nevertheless the book is epoch-making. It finds "that projective geometry, which has no reference to quantity, is necessarily true of any form of externality." "In metrical geometry is an empirical element, arising out of the alternatives of Euclidean and non-Euclidean space."

One of the most pleasing aspects of the universal permanent progress in all things non-Euclidean is the making accessible of the original masterpieces.

The marvelous '*Tentamen*' of Bolyai Farkas, as Appendix to which the 'Science Absolute' of Bolyai János appeared, a book so rare that except my own two copies, I know of no copy on the western continent, a book which has never been translated, a field which has lain fallow for sixty-five years, is now being re-issued in sumptuous quarto form by the Hungarian Academy of Sciences. The first volume appeared in 1897, edited with sixty-three pages of notes in Latin, by König and Réthy of Budapest. Professor Réthy, whom I had the pleasure of meeting in Kolozsvár, tells me the second volume is in press, and he is working on it this summer.

Bolyai Farkas is the forerunner of Helmholtz, Riemann, Lie, though one would scarcely expect it from the poetic exaltation with which he begins his great work. "*Lectori salutem!* Scarce superficially imbued with the rudiments of first principles, of my own accord, without any other end, but led by internal thirst for truth, seeking its very fount, as yet a beardless youth I laid the foundations of this 'Tentamen.'

Only fundamental principles is it proposed here so to present, that Tyros, to whom it is not given to cross on light wings the abyss, and, pure spirits, glad of no original, to be borne up in airs scarce respirable, may, proceeding with firmer step, attain to the heights.

You may have pronounced this a thankless task, since lofty genius, above the windings of the valleys, steps by the Alpine peaks; but truly everywhere are present gordian knots needing swords of giants. Nor for these was this written.

Forsooth I wish the youth by my example warned, lest having attacked the labor of six thousand years, alone, they wear away life in seeking now what long ago was found. Gratefully learn first what predecessors teach, and after forethought build. Whatever of good comes, is antecedent term of an infinite series."

His analysis of space starts with the principle of continuity: spatium est quantitas, est continuum (page 442). This Euclid had used unconsciously, or at least without specific mention; Riemann and Helmholtz consciously. Second comes what he calls the *axiom of congruence* (page 444, §3): "corpus idem in alio quoque loco videnti, quaestio succurrit: *num loca ejusdem diversa aequalia sint?* Intuitus ostendit, aequalia esse."

Riemann: "Setzt man voraus, dass die Koerper unabhaengig vam Ort existieren, so ist das Kruemmungsmass ueberall constant." See also the second hypothesis of Helmholtz.

Third, any point may be moved into any other; the free mobility of rigid bodies. If any point remains at rest any region in which it is may be moved about it in innumerable ways, and so that any point other than the one at rest may recur. If two points are fixed, motion is still possible in a specific way. Three fixed points not costraight prevent all motion (page 446, §5).

Thus we have the third assumption of Helmholtz, combined with his celebrated principle of Monodromy.

Bolyai Farkas deduces from these assumptions not only Euclid but the non-Euclidean systems of his son János, referring to the approximate measurements of astronomy as showing that the parallel postulate is not sufficiently in error to interfere with practice (page 489). This is just what Riemann and Helmholtz afterward did, only by casting off also the assumption of the infinity of space they got also as a possibility for the universe an elliptic geometry, the existence of a case of which independently of parallels was first proven by Bolyai János when he proved spherics independent of Euclid's assumption. So if Sophus Lie had ever seen the 'Tentamen,' he might have called his great investigation the Bolyai Farkas Space Problem instead of the Riemann-Helmholtz Space Problem.

The first volume of the 'Tentamen' as issued by the Hungarian Academy does not contain the famous Appendix. But in 1897 Franz Schmidt, that heroic figure, ever the bridge between János and the world, issued at Budapest the Latin text of the Science Absolute, with a biography of Bolyai János in Magyar, and a Magyar translation of the text by Suták József.

Strangely enough, though the Appendix had been translated into German, French, Italian, English, and even appeared in Japan, yet no Hungarian rendering had ever appeared. It was Franz Schmidt who placed the monument over the forgotten grave of János, only identified because there still lived a woman who had loved him. How in this Magyar edition he rears a second monument. The introduction by Suták is particularly able.

The Russians have honored themselves by the great Lobachévski Prize; why does not that glorious race, the Magyars, do tardy justice to their own genius in a great Bolyai Prize?

One other noble thing the Hungarian Academy of Science has just achieved, the publication in splendid quarto form of the correspondence between Gauss and Bolyai Farkas: (Briefwechsel zwischen Carl Friedrich Gauss und Wolfgang Bolyai). It was again Franz Schmidt who, after long endeavors, at last obtained this correspondence from the Royal Society of Sciences at Göttingen, where Bolyai had sent the letters of Gauss at his death. The correspondence is fitly edited by Schmidt and Staeckel. It gives us a romance of pure science. Gauss was the greater mathematician; Bolyai the nobler soul and truer friend. On April 10, 1816, Bolyai wrote to Gauss giving a detailed account of his son János, then fourteen years old; and unfolding a plan to send János in two years to Göttingen, to study under Gauss. He asks if Gauss will take János into his house, of course for the usual remuneration, and what János shall study meanwhile. Gauss never answered this beautiful and pregnant letter, and never wrote again for sixteen years! Had Gauss answered that letter, Göttingen might now perhaps have to boast a greater than Gauss, for in sheer genius, in magnificent nerve, Bolyai János was unsurpassable, as absolute as his sience of space. But instead he joined the Austrian army, and the mighty genius which should have enriched the transactions of the greatest of learned societies with discovery after discovery in accelerating quickness, preyed instead upon itself, printing nothing but a brief two dozen pages.

Almost to accident the world owes the admirable volumes in which Staeckel and Engel contribute such priceless treasures to the non-Euclidean geometry. An Italian Jesuit, P. Manganotti, discovered that one of his order, the Italian Jesuit Saccheri, had already in 1733 published a series of theorems which the world had been ascribing to Bolyai. Thereupon in 1889 E. Beltrami published in the *Atti della Reale Accademia dei Lincei*, Serie 4, Vol. V., pages 441—448, a note entitled "Un Precursore Italiano di Legendre e di Lobatschewski," giving extracts from Saccheri's book which abundantly proved the claim of Manganotti.

In the same year, 1889, E. d'Ovidio, in the *Torino Atti* XXIV, pages 512—513, called attention to this note in another entitled "Cenno sulla Nota del prof." E. Beltrami: "Un Precursore etc.", expressing the wish that P. Manganotti would by a more ample discussion rescue Saccheri's work from unmerited oblivion. Staeckel says the thought then came to him, whether Saccheri's work were not a link in a chain of evolution, the genesis of the non-Euclidean geometry.

In 1893 at the International Mathematical Congress at Chicago, in the discussion which followed my lecture "Some salient points in the history of non-Euclidean and hyper-spaces," wherein I gave an account of Saccheri with a description of his book and extracts from it, Professor Klein, who had never before heard of Saccheri, and Professor Study of Marburg, mentioned that there had recently been brought to light an old paper of Lámbert's anticipating in points the non-Euclidean geometry, and named in connection therewith Dr. Staeckel. I at once wrote to him and published in the Bulletin of the New York Mathematical Society, Vol., III., pages 79—80, 1893, a note on Lambert's non-Euclidean geometry, mentioning Staeckel's purpose to republish Lambert's paper in the Abhandlungen of the Leipziger Gesellschaft der Wissenschaften. But after this, in January, 1894, Staeckel formed the plan to make of Saccheri and Lambert a book, and associating with him his friend Friedrich Engel, they gave the world in 1895, "Die Theorie der Parallelinien, eine Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometrie." Strengthened by the universal success of this book, they planned two volumes in continuation. Staeckel takes the volume devoted to Bolyai János and his father. It is to begin with a more complete life of the two than has yet appeared, of course from material furnished largely by Franz Schmidt.

Then follows the "Theoria parallelarum" of Bolyai Farkas, interesting as proving that in 1804 Gauss was still under the spell of Euclid.

Then is to follow the Latin text of the immortal Appendix with a German translation. Next comes in German translation selections from the 'Tentamen.' The book concludes with the geometric part of "Kurzer Grundriss," the only one of the Bolyai's works printed originally in German. This volume is nearly published and may be expected in a few weeks. The volume undertaken by Engel has just appeared (1899). It is a German translation of Lobachévski's first published paper (1829) "On the principles of geometry," and also of his greatest

work, "New elements of geometry, with complete theory of parallels." Only from the 'New Elements' can any adequate idea be obtained of the height, the breadth, the depth of Lobachévski's achievement in the new universe of his own creation.

Of equal importance is the fact that Engel's book gives to the world at last a complete, available text-book of non-Euclidean geometry. There is no other to compare with it.

For the history of non-Euclidean geometry we have the admirable chapter X of Loria's pregnant work "Il passato ed il presente delle principali teorie geometriche." This chapter cites about 80 authors, mostly of writings devoted to non-Euclidean geometry.

In my own "Bibliography of hyper-space and non-Euclidean Geometry," in the *Amertcan Journal of Mathematics* (1878), I gave 81 authors and 174 titles. This when reprinted in the Collected Works of Lobachévski (Kazan, 1886) gives 124 authors and 272 titles.

Robert Bonola has just given in the Bollettino di Bibliografia e Storia della Scienze Matematiche (1899) an exceedingly rich and valuable "Bibliografia sui Fondamenti della Geometria in relazione alla Geometria Non-Euclidea," in which he gives 353 titles.

This extraordinary output of human thought has henceforth to be reckoned with. Hereafter no one may neglect it who attempts to treat of fundamentals in geometry or philosophy.

Austin, Texas, August 14, 1899.

NOTE ON THE LOXODROMICS OF THE SPHERE.

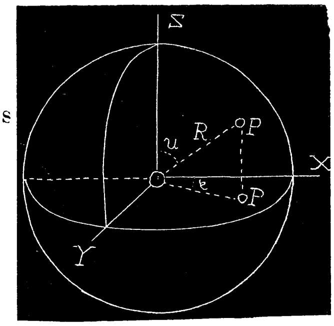
By HERMANN EMCH, of the University of Berne, Switzerland.

1. There is hardly a geometrical problem which from a didactic and practical point of view, and as an application of elementary calculus, is more valuable than that of the loxodromics of the sphere. In Nos. 6-7 of THE AMERICAN MATHEMATICAL MONTHLY (1899), Prof. G. B. M. Zerr proposed the problem to find the length of a N. W.-loxodromic of the earth-surface between the equator and certain parallels and meridians. This note has been prepared in view of this proposition and its value.

2. Let ε be the longitude and u the co-latitude of a point P on the surface of the earth supposed to be spherical and of radius R (Fig. 1).

The Cartesian coördinates x, y, z of P with regard to the planes of the 90°-meridian, the zero-meridian and the equator are, since $OP'=r=R\sin u$,

$$\left.\begin{array}{l} x=R.\sin u.\cos\varepsilon \\ y=R.\sin u.\sin\varepsilon \\ z=R.\cos u \end{array}\right\}\ldots\ldots(1).$$



The square of the linear element of the sphere is

$$ds^2=dx^2+dy^2+dz^2,$$

or since

$$dx=R^2(-\sin u.\sin\varepsilon.d\varepsilon+\cos u.\cos\varepsilon.du),$$
$$dy=R^2(\sin u.\cos\varepsilon.d\varepsilon+\cos u.\sin\varepsilon.du),$$
$$dz=R^2(-\sin u.du),$$

after some calculations,

$$ds^2=R^2.\sin^2 u\left(d\varepsilon^2+\frac{du^2}{\sin^2 u}\right)\ldots\ldots(2).$$

Putting $\int\frac{du}{\sin u}=kx$, $\varepsilon=ky$, or $kx=\log\tan\frac{1}{2}u\ldots\ldots(3)$, $ky=\varepsilon\ldots\ldots(4)$,

where k designates any constant, we have

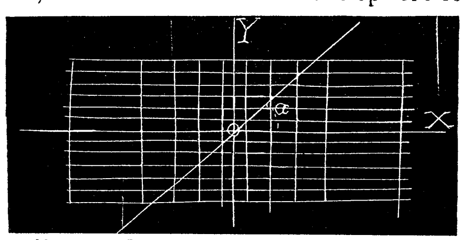
$$ds^2=R^2.k^2.\sin^2 u(dx^2+dy^2).$$

From (2), $\sin^2 u=\frac{4e^{2kx}}{(1+e^{2kx})^2}$, hence $ds^2=R^2.\frac{4k^2.e^{2kx}}{(1+e^{2kx})^2}(dx^2+dy^2)\ldots\ldots(5)$.

3. Considering x and y as Cartesian coördinates of a plane (Fig. 2), it is seen that the surface of the sphere is mapped upon the plane by means of the formulas (2) and (3). The linear element of the XY-plane being $ds'=\sqrt{dx^2+dy^2}$, it follows that the ratio ds/ds' of two corresponding elements on the sphere, and the plane,

$$\frac{ds}{ds'}=\frac{2k.e^{kx}}{1+e^{2kx}}\ldots\ldots(6),$$

for infinitely small variations of x is constant, so that the surface of the sphere is conformally mapped upon the plane.



As it is well known this representation is called Mercator-projection. For constant values for u and e the system of parallels, and meridians, arises.

Parallels if u is constant.

Meridians if ε is constant.

In the Mercator-projection to the parallels and meridians of the globe correspond two systems of parallel lines perperdicular to each other. According to the conformal projection any two lines on the sphere including a certain angle are

projected into two lines including the same angle. Consequently a loxodromic line on the sphere, *i. e.*, a curve which includes constant angles with the meridians is transformed into a straight line, and conversely to a straight line of the Mercator-projection corresponds a loxodromic line on the sphere.

In order to obtain the equation of the loxodromic line assume the two perpendicular lines as Cartesian axes, which correspond to the equator and the new meridian, respectively.

For the equator, $u=\frac{1}{2}\pi$, so that $kx=\log\tan\frac{1}{2}u=\log\tan\frac{1}{4}\pi=0$, or $x=0$, which represents the Y-axis. For the zero-meridian, $\varepsilon=0$, hence $\gamma=0$, so that the X-axis corresponds to the zero-meridian. To a straight line $\gamma=mx$ corresponds the loxodromic line $\varepsilon=m.\log\tan\frac{1}{2}u$, or

$$\tan\tfrac{1}{2}u=e^{\varepsilon/m}\ldots\ldots(7).$$

This loxodromic passes through the point of intersection of the equator and the zero-meridian, and includes an angle α with the meridian whose trigonometric tangent is m. The equation shows that the curve is winding an infinite number of times around the poles.

4. The projection of the loxodromic line upon the plane of the equator is a symmetrical double spiral, whose equation is obtained by substituting the value of $\sin u$ in the expression $r=R.\sin u$.

Now $\tan\frac{1}{2}u=e^{\varepsilon/m}$, from which

$$\sin u=\frac{2e^{\varepsilon/m}}{1+e^{2\varepsilon/m}},\ \text{hence}\ r=\frac{2R.e^{\varepsilon/m}}{1+e^{2\varepsilon/m}},\ \text{or}\ r=\frac{2R}{e^{\varepsilon/m}+e^{-(\varepsilon/m)}}\ldots\ldots(8).$$

From this equation it is seen that the projected curve is symmetrical with regard to the zero-meridian.

5. To find the length of the loxodromic line between the equator and a parallel, for which $u=b$, we have

$$y=x\tan\alpha,\ \text{or}\ y=xm,$$

$$dy=m.dx,$$

$$dy=m.\frac{du}{\sin u},$$

$$ds^2=R^2.\sin^2 u\left(\frac{du^2}{\sin^2 u}+m^2\frac{du^2}{\sin^2 u}\right),$$

$$ds=R.\sqrt{1+m^2}.du,\ \text{and finally,}$$

$$S=R\int_b^{\frac{1}{2}\pi}\sqrt{1+m^2}.du=R\frac{\frac{1}{2}\pi-b}{\cos\alpha}\ldots\ldots(9).$$

This result may be stated in the theorem :

The length of a loxodromic whose inclination with the meridians is α, between two parallels with the latitudes a and b, is equal to the hypothenuse of a right triangle with the difference $(b-a)$ of the latitudes as one side and α as the adjacent angle.

6. If the length of the loxodromic between the equator and the meridian has to be found the element ds must be expressed by the variable ε. The equation of the Mercator-projection of the loxodromic being $\gamma = mx$, or on the sphere $\varepsilon = m\log \tan\frac{1}{2}u$, it follows that

$$d\varepsilon = m.\frac{du}{\sin u},\ \text{or}\ \frac{du}{\sin u} = \frac{d\varepsilon}{m},\ \text{thus}\ ds^2 = R^2\sin^2 u.\left(\frac{d\varepsilon^2}{m^2} + d\varepsilon^2\right),\ \text{or}$$

$$ds^2 = R^2\sin^2 u\left(\frac{1+m^2}{m^2}\right)d\varepsilon^2.$$

Now $\sin^2 u = \frac{e^{2\varepsilon\, m}}{(1+e^{2\varepsilon/m})^2}$, consequently $ds = R.\sqrt{\frac{1+m^2}{m^2}}.\frac{e^{\varepsilon/m}}{1+e^{2\varepsilon\, m}}.d\varepsilon$,

$$\text{and}\ S = R.\int_0^{\varepsilon}\sqrt{\frac{1+m^2}{m^2}}.\frac{e^{\varepsilon/m}}{1+e^{2\varepsilon/m}}.d\varepsilon \ldots\ldots (10).$$

The value of this integral is:

$$S = R.\sqrt{1+m^2}.(\text{arc}\ \tan e^{\varepsilon/m} - \tfrac{1}{4}\pi) \ldots\ldots (11).$$

The length of the loxodromic line between $2(k-1)\pi$ and $2k\pi$, or of the kth winding, is

$$S_k = R.\sqrt{1+m^2}(\text{arc}\ \tan e^{2k\pi\, m} - \text{arc}\ \tan e^{2(k-1)\pi/m}).$$

This value decreases as k increases and vanishes for $k = \infty$. For $k = 1, 2, 3, \ldots\ldots \infty$, $S = S_1 + S_2 + S_3 + \ldots\ldots$ ad infinitum, and appears as a convergent series, as it may easily be verified.

7. Numerical examples concerning a loxodromic, having an inclination with the meridian of 45°.

Radius of earth = 3956 miles.

A. Length of loxodromics from equator to parallels 30, 45, 60, 90, in miles.

Latitude.	Length of Loxodromic.
0	0
30	2929
45	4394
60	5859
90	8788

B. Length of loxodromics from equator and zero-meridian to meridians 90, 180, 270, 360 in miles.

Longitude.	Length of Loxodromic.
0	0
90	3247
180	4152
270	4344
360	4384

The entire length of the loxodromic is $S=\sqrt{(2)}\frac{1}{2}\pi.R=8788$ miles, which is obtained by putting in (9) $b=0$, or in (11) $\varepsilon=\infty$. This result coincides with the one obtained in the first table, where the length of the loxodromic for the latitude of 90° is also 8788 miles.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

114. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Does it pay a $4-carpenter using a dozen four-penny nails per minute, to pick up a dropped nail? At this rate, should twenty-penny nails be picked up?

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

The price of four-penny nails, at the present time, is 5 cents per pound. Assume that there are 200 nails to the pound, and that it takes the carpenter 10 seconds to pick up a nail.

The value of a nail is $\frac{5}{200}$ of a cent, or $\frac{1}{40}$ of a cent.

If we assume that the carpenter gets $4.00 per day, and works 10 hours in a day, his wages is 40 cents per hour, or $\frac{1}{90}$ of a cent per second.

Hence, 10 seconds, the time required to pick up a nail, is worth $\frac{1}{9}$ of a cent.

Hence, since the value of the nail picked up is only $\frac{1}{40}$ of a cent, it does not pay the carpenter to pick up the nail, he losing thereby $\frac{1}{9}-\frac{1}{40}$ or $\frac{31}{360}$ of a cent.

It would not pay to pick up twenty-penny nails at the same rate.

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics in Decorah Institute, Decorah, Ia.

Where shall a pole 120 feet high be broken so that the top may rest on the ground 40 feet from the foot? (Solve by arithmetic.)

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

A. 40=base, and 120=sum of altitude and hypothenuse of a right-angled triangle.

$(120^2 \pm 40^2) \div 120 = 133\frac{1}{3}$ and $106\frac{2}{3}$, which are, respectively, two times the hypothenuse, and two times the altitude.

$\therefore$ Hypothenuse$=66\frac{2}{3}$ feet, and altitude$=53\frac{1}{3}$ feet.

$\therefore$ The pole must be broken $53\frac{1}{3}$ feet from its foot.

B. 40=base and 120=sum of altitude and hypothenuse of a right-angled triangle.

Since one of the sides and the sum of the other two sides are rational, each of the sides must be rational.

Also, prime, integral right triangles are the basis of all composite and fractional rational right triangles.

We observe that the base is one-third of the sum of the other two sides. This is the case with *one right triangle of prime, integral sides*, of which the base =3, altitude=4, and hypothenuse=5.

$40 = 13\frac{1}{3}$ times 3. $\therefore$ The altitude and hypothenuse of the required right triangle are, respectively, $13\frac{1}{3} \times 4$, and $13\frac{1}{3} \times 5$, or $53\frac{1}{3}$ and $66\frac{2}{3}$.

$\therefore$ The pole must be broken $53\frac{1}{3}$ feet from its foot.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; ELMER SCHUYLER, Annapolis, Md.; P. S. BERG, B. Sc., Larimore, N. D.; H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa., and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$AB=120$ feet, $BC=40$ feet. Since $CD=AD$, $\angle CAD=DCA$.

Hence (geometrically), construct a right triangle with base 40 and perpendicular 120, and join AC. At C, make an angle equal to CAB, and this will give D, the required point.

Algebraically: $AD=CD=x$, $BD=y$, and we have at once $x+y=120$ ......(1), $x^2-y^2=1600$......(2). Dividing (2) by (1) we have $x-y=13\frac{1}{3}$.

Whence $x=66\frac{2}{3}$, and $y=53\frac{1}{3}$.

III. Solution by CHARLES CARROLL CROSS, Whaleyville, Va.

A rule, which can easily be established by geometry for solving such problems is as follows: Divide the difference of the squares of the height of the pole and the distance on the ground by two times the height, which will give the part standing. Thus height$=(120^2-40^2)/2\times120=53\frac{1}{3}$ feet.

[Is this an arithmetical solution? C. C. C.]

116. Proposed by J. O. MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.

Two candles are of the same length. The one is consumed uniformly in 4 hours, and the other in 5 hours. If the candles are lighted at the same time, when will one be three times as long as the other?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Ia.; D. G. DORRANCE, Jr., Camden, N. Y.; ELMER SCHUYLER, United States Naval Academy, Annapolis, Md., and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let x=the time in hours.

Then, at the end of x hours, the unconsumed portions of the candles will be $\frac{4-x}{4}$ and $\frac{5-x}{5}$ of their respective lengths.

$\therefore \frac{5-x}{5} \div \frac{4-x}{4} = 3.$

$\therefore x = 3\frac{7}{11}$ hours, the time required.

COROLLARY. Putting a and b=the respective times of consumption, $a>b$, and n=the ratio of unconsumed lengths, we have $\frac{a-x}{a} \div \frac{b-x}{b} = n$.

$\therefore x = \frac{ab(n-1)}{an-b}$. GRUBER.

Also solved by *W. F. SHAW*, Austin, Tex.

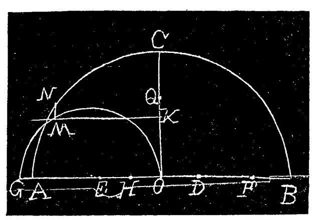
GEOMETRY.

116. Proposed by P. S. BERG, A. M , Superintendent of Schools, Larimore, N. D.

Inscribe by rule and compass a regular heptadecagon.

I. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

The following construction is found on page 95, Vol. IV, of the *Analyst*. This construction is credited to Leybourne's *Mathematical Repository*, 1818.



Let ACB be a semi-circle. Draw the radius CO perpendicular to the diameter AB; on OC and OB take QO equal to the half, and OD equal to the eighth part of the radius; make DE and DF each equal to DQ, and EG and FH, respectively, equal to EQ and FQ; take OK a mean proportional between OH and OQ, and through K draw KM parallel to AB, meeting the semi-circle described on OG in M; and draw MN parallel to OC, cutting the given circle in N. Then the arc AN is the seventeenth part of the whole circumference. The above figure is not drawn to scale.

For a complete and detailed construction see Klein's *Famous Problems in Elementary Geometry*, translated from the German by Professors Beman and Smith, pages 24—41. An analytic solution due to Ampere may be seen in Casey's *Elements of Euclid*. See also THE AMERICAN MATHEMATICAL MONTHLY, Vol. I, page 376, an article by Dr. L. E. Dickson.

II. An approximate construction by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Draw $AB=6\frac{1}{4}$ times the radius. Draw AC. On AC lay off seventeen equal portions. Let AD, DE, EF, and so on, be the equal portions. Join CB and draw DG, EH, FK, etc., parallel to BC. Then AG or GH or HK, etc., are sides of the regular heptadecagon required.

Let a=radius, s=side.

$\therefore$ $s=2a\sin\frac{1}{17}\pi=.3675a$.

$6\frac{1}{4}a=6\ 25a$. $6.25a\div17=.3676a$.

This shows that the above method is a very close approximation.

Prof. Cooper D. Schmitt did not give a construction but gave several references.

117. Proposed by GUY B. COLLIER, Schenectady, N. Y.

If (x', y') and (x'', y'') are the extremities of a pair of conjugate diameters whose eccentric angles are φ' and φ, show that $\varphi'+\varphi=90°$; given $(x'\ y')=(a\sec\varphi', b\tan\varphi')$. [From Nichols' *Analytical Geometry*.]

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.

Let OQ be the diameter conjugate to OP. Then OQ meets the conjugate hyperbola in Q, R. Let $OA=a$, $OB=b$. Draw PD, QF perpendicular to OD, OF, respectively. Draw DG tangent to the circle with radius OA, FH tangent to the circle with radius OB. Draw OG, OH perpendicular to DG, FH, respectively.

Then $PD=y'$, $OD=x'$, $OF=y''$, $FQ=x''$, $\angle GOD=\varphi'$, $\angle HOA=\varphi$, $x'=a\sec\varphi'$, $y'=b\tan\varphi'$, $x''=a\cot\varphi$, $y''=b\operatorname{cosec}\varphi$. Also $x''=(a/b)y'$, $y''=(b/a)x'$.

$\therefore$ $a\cot\varphi=(a/b)b\tan\varphi'$. $\therefore$ $\cot\varphi=\tan\varphi'$.

$\therefore$ $\varphi=\frac{1}{2}\pi-\varphi'$. $\therefore$ $\varphi+\varphi'=\frac{1}{2}\pi$.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professsor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

In the last line of the problem, for $b\tan\varphi$ put $b\tan\varphi'$.

It is well known that $x'=a\sec\varphi'$, $y'=b\tan\varphi'$ represents a point on the hyperbola $x^2/a^2-y^2/b^2=1$. Also, (x'', y'') is on the conjugate hyperbola, $x^2/a^2-y^2/b^2=-1$, requiring that $x''=a\tan\varphi$, $y''=b\sec\varphi$.

But $x''=(a/b)y'=a\tan\varphi'$, $y''=(b/a)x'=b\sec\varphi$.

$\therefore$ $a\tan\varphi'=a\tan\varphi$, and $b\sec\varphi'=b\sec\varphi$, from either of which $\varphi'=\varphi$.

Of a half dozen books on Analytic Geometry, selected somewhat at random, only one explains this matter with a desirable degree of clearness and correctness.

III. Solution by GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri, School of Mines and Metallurgy, Rolla, Mo.

Let φ' be the eccentric angle of x', y' with respect to the transverse axis.

φ'', eccentric angle of x'', y'' with respect to transverse axis.

φ''', eccentric angle of x'', y'' with respect to conjugate axis.

Then since the conjugate hyperbola bears the same relation to the conjugate axis that the given hyperbola does to the transverse, we may write $x''=a\tan\varphi'''$, $y''=b\sec\varphi'''$.

But $x''/a=y'/b$, $y''/b=x'/a$.

Hence $x''=a\tan\varphi'$, $y''=b\sec\varphi'$.

Therefore $\varphi'=\varphi'''$. But $\varphi''+\varphi'''=90^\circ$.

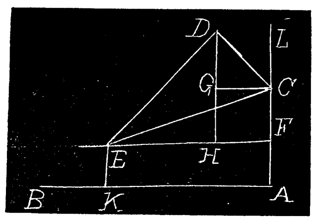
Then $\varphi'+\varphi''=90^\circ$. Q. E. D.

118. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

A picture b feet long hangs on a wall at an inclination θ to the wall, with its base a feet from the floor. How far from the wall should an admirer sit to see it to the best advantage, supposing the light to be equally distributed throughout the room.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass., and J. M. BANDY, A. M., Ph. B., Greensboro, N. C.

Let E be the position of the eye, $EK=FA=c$, $AC=a$, CD, the length of the picture$=b$, $FE=x$, $\angle LCD=\theta$, $\angle FED=\beta$, $\angle FEC=\gamma$, $\angle CED=\lambda$.



The picture is seen to the best advantage when λ is a maximum.

$$\tan\lambda=\frac{\tan\beta-\tan\gamma}{1+\tan\beta\tan\gamma}.$$

$CG=b\sin\theta$, $DG=b\cos\theta$, $CF=a-c$, $EH=x-b\sin\theta$, $DH=a-c+b\cos\theta$.

$$\tan\beta=\frac{a-c+b\cos\theta}{x-b\sin\theta},\ \tan\gamma=\frac{a-c}{x}$$

$$\therefore\ \tan\lambda=\frac{bx\cos\theta+b\sin\theta(a-c)}{x^2-bx\sin\theta+(a-c)(a-c+b\cos\theta)}=\text{maximum}.$$

Differentiating and collecting terms we get,

$$x^2+2x(a-c)\tan\theta-(a-c)^2-b(a-c)\sec\theta=0.$$

$$\therefore\ x=\sqrt{(a-c)^2\sec^2\theta+b(a-c)\sec\theta}-(a-c)\tan\theta.$$

If $c=0$ and $\theta=0$, $x=\sqrt{a(a+b)}$; if $c=0$ and $\theta=90^\circ$, $x=\frac{1}{2}b$; if $c=0$, $\theta=0$, and $b=\frac{5}{12}a$, $x=\frac{13}{12}a$.

Also solved, with same result, by *B. F. SINE*, *ELMER SCHUYLER*, *CHARLES C. CROSS*, and *P. S. BERG*.

Walter H. Drane should have received credit for a solution of problem 115 in the last issue.

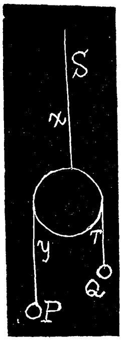
MECHANICS.

84. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Two weights P and Q are fastened by a weightless string that is strung over a single movable pulley. P is greater than Q. The weight of the pulley is $2R$. Find the tension of the string, (1) when the friction of the string on the pulley is neglected, (2) when it is considered.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1) Let S be the force acting upwards on the pulley, T the tension required, x, y as in the figure. The velocity of the pulley is dx/dt, the moving force $2R+2T-S$. The equation of motion, therefore, is



$$\frac{2R}{g}.\frac{d^2x}{dt^2}=2R+2T-S\ldots\ldots(1).$$

Similarly for weights P and Q we have

$$\frac{P}{g}\left(\frac{d^2y}{dt^2}+\frac{d^2x}{dt^2}\right)=P-T\ldots\ldots(2).$$

$$\frac{Q}{g}\left(\frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}\right)=T-Q\ldots\ldots(3).$$

Eliminating d^2y/dt^2 between (2) and (3) we get

$$\frac{d^2x}{dt^2}=\frac{2PQg-Tg(P+Q)}{2PQ}\ldots\ldots(4).$$

Eliminating d^2x/dt^2 between (1) and (4) we get

$$T=\frac{PQS}{2PQ+R(P+Q)}.$$

(2) In this case we will regard the pully as perfectly rough and disregard friction on the axle of the pulley. Three other cases are possible, as follows: Smooth axle, pulley imperfectly rough; rough axle, pully imperfectly rough; rough axle, pulley perfectly rough.

Let T'=tension caused by P, T''=tension caused by Q, θ=angle through which the pulley turns, a=radius of pulley, $k^2=\frac{1}{2}a^2$=radius of gyration.

Then we have

$$\frac{d^2x}{dt^2}=\frac{(2R+T'+T''-S)g}{2R}\ldots\ldots(5).$$

$$\frac{d^2y}{dt^2}+\frac{d^2x}{dt^2}=\frac{(P-T')g}{P}\ldots\ldots(6).\quad \frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}=\frac{(T''-Q)g}{Q}\ldots\ldots(7).$$

$$\frac{2Rk^2}{g}\cdot\frac{d^2\theta}{dt^2}=a(T'-T'')\ldots\ldots(8).\quad a\theta=y\ \ldots\ldots(9).$$

From (9), $ad^2\theta/dt^2=d^2y/dt^2$. This in (8), with $k^2=\frac{1}{2}a^2$, gives

$$\frac{d^2y}{dt^2}=\frac{(T'-T'')g}{R}\ldots\ldots(10).$$

Eliminating d^2y/dt^2 and d^2x/dt^2 from (5), (6), (7), and (10), we get

$$(3P+2R)T'-PT''=PS.\quad QT'+(2R-3Q)T''=4RQ-QS.$$

$$\therefore T'=\frac{2PQ(R-S)+PRS}{P(3R-4Q)+R(2R-3Q)}.\quad T''=\frac{2PQ(3R-S)+QR(4R-S)}{P(3R-4Q)+R(2R-3Q)}.$$

II. Solution by **WILLIAM HOOVER, A. M., Ph. D.,** Professsor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let T, T' be the two tensions in the parts of the string to which P and Q are respectively attached, μ=the coefficient of friction between the string and pulley, a, k, the radius and radius of gyration, $m=2R/g$=the mass of the pulley, θ=the angle through which the pulley has rotated in the time t from the beginning of motion, s=the distance of the ascending weight above the earth at the time t.

For the motion of the weights, vertically,

$$\frac{P}{g}\frac{d^2s}{dt^2}=(P-T)\ldots\ldots(1),\quad \frac{Q}{g}\frac{d^2s}{dt^2}=(T'-Q)\ldots\ldots(2).$$

For the motion of the pulley,

$$mk^2\frac{d^2\theta}{dt^2}=a(T-T')\ldots\ldots(3).$$

Eliminating d^2s/dt^2 from (1) and (2), $PT'+QT=2PQ\ldots\ldots(4)$.
Again, from the theory of friction, $T'=Te^{-\mu\pi}=cT\ldots\ldots(5)$.
Substituting in (4),

$$T=\frac{2PQ}{cP+Q}\ldots\ldots(6),\quad T'=\frac{2cPQ}{cP+Q}\ \ldots\ldots(7).$$

(6) and (7) in (3) gives

$$mk^2\frac{d^2\theta}{dt^2}=\frac{2a(1-c)PQ}{cP+Q}\ldots\ldots(8).$$

Integrating (8), supposing $d\theta/dt=0$, when $t=0$,

$$mk^2\frac{d\theta}{dt}=\frac{2a(1-c)PQ}{cP+Q}t\ldots\ldots(9).$$

If there be no friction, as in (1) of the problem, $\mu=0$, $c=1$,

$$T=T'=\frac{2PQ}{P+Q}\ldots\ldots(10).$$

and (9) shows that there would be no rotation of the pulley.

85. Proposed by **WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $\sqrt{\frac{ng}{a}}$, and a particle is projected from its lowest point with such velocity that it can just reach the highest point; prove that the time of describing the first quadrant is $\sqrt{\frac{a}{(n+1)g}}\log(\sqrt{n+2}+\sqrt{n+1})$.

I. Solution by the PROPOSER.

Let $a\theta$ be the arc over which the particle has passed in any time t from the beginning of motion, R=reaction of the curve, g=the acceleration of gravity, and put $\sqrt{\frac{ng}{a}}=\omega$.

Resolving vertically and horizontally,

$$a\frac{d^2(\cos\theta)}{dt^2}=g-R\cos\theta\ldots\ldots(1),\quad a\frac{d^2(\sin\theta)}{dt^2}-\omega^2a\sin\theta=-R\sin\theta\ldots\ldots(2).$$

Eliminating R,

$$a\frac{d^2\theta}{dt^2}-a\omega^2\sin\theta\cos\theta=-g\sin\theta\ldots\ldots(3).$$

Integrating (3),

$$\frac{d^2\theta}{dt^2}=\frac{2g}{a}\cos\theta-\omega^2\cos^2\theta+C\ldots\ldots(4).$$

When $\theta=0$, $\frac{d\theta}{dt}=\frac{4g}{a}$; $\therefore$ $C=\frac{2g}{a}+\omega^2$, and (4) becomes

$$\frac{d\theta^2}{dt^2}=\frac{2g}{a}\cos\theta-\omega^2\cos^2\theta+\frac{2g}{a}+\omega^2\ldots\ldots(5).$$

From (5), $dt=\dfrac{d\theta}{\sqrt{\left(\frac{2g}{a}\cos\theta-\omega^2\cos^2\theta+\frac{2g}{a}+\omega^2\right)}}\ldots\ldots(6).$

Let $\cos\theta=\dfrac{2-y}{3+y}\ldots\ldots(7)$, the limit of θ being 0 and $\frac{1}{2}\pi$, and of y, 2 and $-\frac{1}{2}$.

From (7), $d\theta=\dfrac{\sqrt{5}\,dy}{(3+y)\sqrt{(1+2y)}}\ldots\ldots(8).$

Substituting (7) and (8) in (6), and reducing,

$$dt=-\frac{dy}{\sqrt{\left[\frac{6g}{a}+\omega^2+2\left(\frac{7g}{a}+\omega^2\right)y+4\left(\frac{g}{a}+\omega^2\right)y^2\right]}}$$

negative, since t increases as y decreases.

Integrating between the above limits for y,

$$t=\frac{1}{\sqrt{\left(\frac{g}{a}+\omega^2\right)}}\log\sqrt{\left\{\frac{\sqrt{\left(\frac{2g}{a}+\omega^2\right)}+\sqrt{\left(\frac{g}{a}+\omega^2\right)}}{\sqrt{\left(\frac{2g}{a}+\omega^2\right)}-\sqrt{\left(\frac{g}{a}+\omega^2\right)}}\right\}}$$

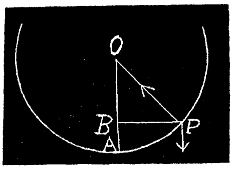
$$=\sqrt{\left(\frac{a}{(n+1)g}\right)}\log[\sqrt{(n+2)}+\sqrt{(n+1)}.$$

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let $OB=z$, $BP=r$, $\angle BOP=\theta$, $PO=a$, ω=angular velocity, $s=a\omega$. The equations of motion are

$$d^2r/dt^2-r\omega^2=R(dz/ds),\quad d^2z/dt^2=g-R(dr/ds),$$

but $r=a\sin\theta$, $z=a\cos\theta$, $ds=ad\theta$.



$\therefore\ a\cos\theta(d^2\theta/dt^2)-2a\omega^2\sin\theta=-R\sin\theta\ldots\ldots$ (1).

$a\sin\theta(d^2\theta/dt^2)+a\omega^2\cos\theta=-R\cos\theta-g\ldots\ldots(2).$

Eliminating R between (1) and (2),

$$a(d^2\theta/dt^2)=a\omega^2\sin\theta\cos\theta-g\sin\theta\ldots\ldots(3).$$

Integrating (3), $a(d\theta/dt)^2=a\omega^2\sin^2\theta+2g\cos\theta+C$.
Since $d\theta/dt=0$ when $\theta=\pi$, $C=2g$, also $\omega^2=ng/a$.
$\therefore\ a(d\theta/dt)^2=ng\sin^2\theta+2g(1+\cos\theta)$.

$$\therefore\ t=\sqrt{\frac{an}{g}}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(n+1)^2-(1-n\cos\theta)^2}}=\tfrac{1}{2}\sqrt{\frac{a}{g}}\int_0^{\frac{1}{2}\pi}\frac{\sec\frac{1}{2}\theta d\theta}{\sqrt{1+n\sin^2\frac{1}{2}\theta}}.$$

$$\therefore\ t=\tfrac{1}{2}\sqrt{\frac{a}{g}}\int_0^{\frac{1}{2}\pi}\frac{\sec^2\frac{1}{2}\theta d\theta}{\sqrt{1+(n+1)\tan^2\frac{1}{2}\theta}}.$$

Let $\tan\frac{1}{2}\theta=\sqrt{\dfrac{1}{n+1}}\tan\varphi$.

$$\therefore\ t=\sqrt{\frac{a}{(n+1)g}}\int_0^{\tan^{-1}\sqrt{n+1}}\sec\varphi d\varphi=\tfrac{1}{2}\sqrt{\frac{a}{(n+1)g}}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)_0^{\tan^{-1}\sqrt{n+1}}$$

$$\therefore\ t=\tfrac{1}{2}\sqrt{\frac{a}{(n+1)g}}\log\left(\frac{\sqrt{(n+2)}+\sqrt{(n+1)}}{\sqrt{(n+2)}-\sqrt{(n+1)}}\right)=\sqrt{\frac{a}{(n+1)g}}\log[\sqrt{n+2}+\sqrt{n+1}]$$

III. Solution by **GEORGE R. DEAN, A. M.,** Professor of Mathematics, University of Missouri, School of Mines and Metallurgy, Rolla, Mo.

(1) Lagrange's equations for a point in space are

$$m\rho''-m\rho(\theta'^2+\sin^2\theta\varphi'^2)=R,$$

$$m\frac{d}{dt}(\rho^2\theta')-m\rho^2\varphi'^2\sin\theta\cos\theta=\rho\Theta,$$

$$m\frac{d}{dt}(\rho^2\sin^2\theta\varphi')=\rho\Phi\sin\theta\ ;$$

where ρ, θ, φ are the polar coördinates, and R, Θ, Φ are the components of the impressed force along and normal to the radius vector. Besides these we have the geometrical equations $\rho=a$ and $\varphi'=c$. Also $R=mg\cos\theta$, $\Theta=mg\sin\theta$.

Replacing the first and last equations by the geometrical equations and eliminating ρ and φ', we find

$$m\frac{d}{dt}(a^2\theta')-ma^2c^2\sin\theta\cos\theta=mga\sin\theta$$

Removing factor ma, multiplying by $d\theta/dt$, and integrating,

$$a\left(\frac{d\theta}{dt}\right)^2-ac^2\sin^2\theta=-2g\cos\theta+K.$$

When $\theta=0$, $d\theta/dt=0$; hence $K=2g$, and we have

$$a\left(\frac{d\theta}{dt}\right)^2=2g(1-\cos\theta)+ac^2\sin^2\theta.$$

Putting $ac^2=ng$, and integrating between limits $\theta=\pi$, $\theta=\frac{1}{2}\pi$, we find

$$t=\sqrt{\frac{a}{(n+1)g}}\log(\sqrt{n+2}+\sqrt{n+1}).$$

(2) The energy of the particle relative to the tube is $\frac{1}{2}m\ a^2\theta'^2$. This is due to two causes, gravity and rotation. Hence

$$\tfrac{1}{2}m\ a^2\theta'^2=amg(1-\cos\theta)+\tfrac{1}{2}m\ a^2\varphi'^2\sin^2\theta,$$

which is the same as the first integral second equation of the Lagrangian groups.

AVERAGE AND PROBABILITY.

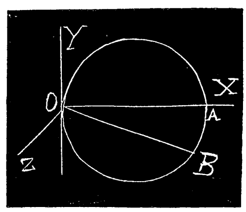
74. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

From a point in the circumference of a circular field a projectile is thrown at random with a given velocity which is such that the diameter of the field is equal to the greatest range of the projectile. Find the chance of its falling into the field. [From Byerly's *Integral Calculus*, page 209.]

I. Solution by the PROPOSER.

It is easily seen that if α, the angle of projection, has a value from $\frac{1}{4}\pi-\frac{1}{2}\phi$ to $\frac{1}{4}\pi+\frac{1}{2}\phi$, the projectile will fall beyond B.

The unfavorable chance is



$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{4}\pi-\frac{1}{2}\phi}^{\frac{1}{4}\pi+\frac{1}{2}\phi}\cos\alpha\, d\alpha\, d\phi=\frac{2}{\pi}(\sqrt{2}-1).$$

Since half of the projectiles will fall on the left of the YZ plane, the favorable chance is $\frac{1}{2}-(2/\pi)(\sqrt{2}-1)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let φ=angle of projection, $OA=2a$, $\angle AOB=\theta$.

$\therefore$ Range$=(v^2/2g)\sin2\varphi=OB=2a\cos\theta$, when the projectile falls within the field.

The range is greatest when $\theta=45°$, and then according to the conditions of the problem $v^2/2g=2a$.

$\therefore$ $2a\sin2\varphi=2a\cos\theta$, or $\sin2\varphi=\cos\theta$.

$\therefore$ $\sin\varphi=\frac{1}{2}(1+\cos\theta)^{\frac{1}{2}}\pm\frac{1}{2}(1-\cos\theta)^{\frac{1}{2}}$.

Therefore the projectile will fall without the circle, if $\sin\varphi$ is less than $\frac{1}{2}(1+\cos\theta)^{\frac{1}{2}}-\frac{1}{2}(1-\cos\theta)^{\frac{1}{2}}$; but will fall within if $\sin\varphi$ is greater than $\frac{1}{2}(1+\cos\theta)^{\frac{1}{2}}+\frac{1}{2}(1-\cos\theta)^{\frac{1}{2}}$.

If all possible directions are equally probable, the chance of the projectile falling within the circle is $1-(1-\cos\theta)^{\frac{1}{2}}=1-\sqrt{2}\sin\frac{1}{2}\theta$.

Hence the required chance is

$$p=\frac{\int_0^{\frac{1}{2}\pi}(1-\sqrt{2}\sin\frac{1}{2}\theta)d\theta}{\int_0^{\pi}d\theta}=\tfrac{1}{2}-(2/\pi)(\sqrt{2}-1).$$

III. Solution by GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.

The diameter of the field is v^2/g, and the range must not exceed $(v^2/g)\cos\phi$. The elevation may vary from 0° to θ_1, and from $\frac{1}{2}\pi-\theta_1$ to $\frac{1}{2}\pi$ for each value of ϕ, where θ_1 is determined by $\sin 2\theta_1=\cos\phi$, ϕ the azimuth of the projectile measured from the diameter, θ the elevation of the gun, v the velocity of projection, and g the intensity of gravity.

The surface-element of the enveloping hemisphere whose radius is R or v^2/g is $R^2\cos\theta d\theta d\phi$.

Using only one-half the field and one-fourth of the sphere the required chance is

$$\frac{R^2\int_0^{\frac{1}{2}\pi}\int_0^{\theta_1}\cos\theta d\theta d\phi+R^2\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta_1}^{\frac{1}{2}\pi}\cos\theta d\theta d\phi}{\pi R^2}$$

$$\text{or } \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}(1-\sqrt{2}\sin\tfrac{1}{2}\phi)d\phi,\ \text{or } \tfrac{1}{2}-\frac{2}{\pi}(\sqrt{2}-1).$$

Solved in a similar manner by *L. C. WALKER*. Solved with different results by *HENRY HEATON*, and *P. H. PHILBRICK*.

75. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the mean area of all plane rectilineal right triangles having a constant perimeter p.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let x and y be the base and altitude, respectively.

Then $x+y+\sqrt{(x^2+y^2)}=p$, the perimeter .....(1).

$$\therefore x=\frac{p^2-2py}{2(p-y)}.\quad \therefore \text{Area}=\frac{p}{4}\frac{(p-2y)y}{p-y}.$$

The limits of y are $y=0$ to $y=x$. When $y=x$ in (1) we get $y=p/(2+\sqrt{2})=y'$.

$$\therefore \triangle=\frac{p}{4}\,\frac{\displaystyle\int_0^{y'}\frac{(p-2y)y}{p-y}dy}{\displaystyle\int_0^{y'}dy}=\frac{2+\sqrt{2}}{4}\int_0^{y'}(2y+p-\frac{p^2}{p-y})dy$$

$$=\frac{2+\sqrt{2}}{4}\left[\frac{1}{(2+\sqrt{2})^2}+\frac{1}{2+\sqrt{2}}+\log\left(\frac{1+\sqrt{2}}{2+\sqrt{2}}\right)\right]p^2.$$

$$=\frac{p^2}{4}\left[\frac{1}{2+\sqrt{2}}+1+(2+\sqrt{2})\log\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$=\frac{p^2}{8}\left[4-\sqrt{2}-(2+\sqrt{2})\log 2\right].$$

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and L. C. WALKER, Associate Professor of Mathematics, Leland Stanford University, Palo Alto, Cal.

Denote the base of the right triangle by x, and the perpendicular by y.

Then the hypotenuse is $p-x-y$, and from $(p-x-y)^2=a^2+y^2$, we obtain

$y=\dfrac{p^2-2px}{2(p-x)}$. $\therefore$ The area$=\dfrac{p}{4}\dfrac{(p-2x)x}{p-x}$, and the average area is

$$\frac{p}{4}\int_0^{\frac{1}{2}p}\frac{p(p-2x)x}{p-x}dx\div\int_0^{\frac{1}{2}p}dx=\tfrac{1}{2}\int_0^{\frac{1}{2}p}\frac{p(p-2x)x}{p-x}dx=\tfrac{1}{2}\int_0^{\frac{1}{2}p}\left(2x+p-\frac{p^2}{p-x}\right)dx$$

$$=\tfrac{1}{8}p^2(3-4\log 2).$$

NOTE.—The difference in the results of these two solutions is due to the different limits assumed for the variable. It seems that the proper limits to assume for the variable x is 0 and $\frac{1}{2}p$, for in this way, and in this way only, do we get the totality of triangles according to our assumed law of distribution of the triangles. The law of distribution tacitly assumed in both solutions is that the number of triangles is proportional to the base of the triangle. Had some other law of distribution been assumed different results from either of the above would have been obtained. The problem is stated in the indefinite form because the law of distribution is not stated. See Dr. E. H. Moore's Note on Mean Values, Vol. II, page 303, of THE AMERICAN MATHEMATICAL MONTHLY. ED. F.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

120. Proposed by ELMER SCHUYLER, United States Naval Academy, Annapolis, Md.

How many balls 1 inch in diameter can be put into a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

121. Proposed by PAUL ROULET, A. M., Professor of Mathematics, Fairmount College, Wichita, Kan.

Three men, named Adams, Morris, and Stoughton, with their sons, Edward, Nathan, and Walter, have each a piece of land in the form of a square. Mr. Adams' piece is 23 rods longer on each side than Nathan's, and Mr. Stoughton's piece is 11 rods longer on each side than Edward's. Each father has 63 square rods of land more than his son. Which of these persons is father and son, respectively?

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

ALGEBRA.

109. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

$\dfrac{1}{\sqrt{(x+1)}}+\dfrac{1}{\sqrt{(x-1)}}=\dfrac{1}{\sqrt{(x^2-1)}}$; find value of x satisfying this equation.

110. Proposed by J. C. CORBIN, Pine Bluff, Ark.

I find the annexed problem in a secular newspaper:

Put down any sum of pounds, shillings and pence under £11, taking care that the number of pence is less than the number of pounds. Reverse this sum, putting pounds in the place of pence, and subtract from original amount. Again reverse this remainder and add. The result in all cases will be £12 18*s* 11*d*, neither more nor less, whatever the amount with which we start.

Will some of the MONTHLY's contributors verify and explain or disprove it?

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

GEOMETRY.

128. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

Given $F=\triangle^{n-1}\div(n-1)!.\triangle_1.\triangle_2\dots\dots\triangle_n$ where $\triangle$=the determinant $(a_1b_2c_3\dots\dots k_n)$ and $\triangle_1\triangle_2\dots\dots\triangle_n$ are the minors of the elements of the nth column; and a_m, b_m, $c_m\dots\dots$etc., $(m=1, 2, 3\dots\dots n)$ are the coefficients of n given equations containing $n-1$ variables. Show (1) that $n=3$, F=the area of a triangle, and (2) if $n=4$, F=the volume of the tetrahedron.

129. Proposed by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that at no point of an ellipse will the circle of curvature pass through the center, if the ecentricity be less than $\frac{1}{2}\sqrt{2}$.

130. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

If the points x, y, z divide the strokes $c—b$, $a—c$, $b—a$, in the same ratio r, and the triangles x, y, z and a, b, c are similar, either $r=1$ or both triangles are equilateral. [From Harkness and Morley's *Introduction to the Theory of Functions*, page 26.]

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

CALCULUS.

99. Proposed by L. C. WALKER, Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

The axis of three equal right circular cylinders intersect at right angles. Find the volume of the solid common to all.

100. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

What is the volume bounded by the surface generated by the circumference of a circle whose diameter is the hypotenuse of a right-angled triangle whose base is b and altitude a, the plane of the circle being perpendicular to the plane of the triangle, the triangle and circle being rigidly connected, and the triangle revolving about its altitude a as an axis?

101. Proposed by WILLIAM FRED FLEMMING, Denison, Tex.

A 24-inch joint of 6-inch stove pipe is compressed at one end to make it fit over an elliptical opening in a stove (for the escape of the smoke). The ellipse has a major axis of 8 inches. What reduction is there in the solid contents of the stove pipe, assuming that its compressed shape may be generated by a 6-inch circle which passes uniformly from one end to the other and perpendicular to the axis of the pipe?

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

MECHANICS.

99. Proposed by G. B. M. ZERR. A. M., Ph. D.. Professor of Mathematics and Science, Chester High School, Chester, Pa.

In a triangle ABC, base$=b$, area$=\triangle$, the principal moments of inertia at the centroid are $\frac{1}{72}m[a^2+b^2+c^2\pm\sqrt{(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)}]$ and the principal axes at this point make with the base AC an angle θ given by

$$\tan2\theta=\frac{4(c^2-a^2)\triangle}{(a^2-c^2)^2-b^2(a^2+c^2)+2b^4}.$$

100. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A man, riding a bicycle, runs through a puddle of water and a bit of mud is thrown from the rear wheel and alights on the crown of his hat. Supposing the wheel 28 inches in diameter, that the man's head is 6 feet above ground, that the saddle is one foot in front of the rear wheel, and that the mud left the wheel at a point 30° from highest point of wheel, how long will it take a man to ride a mile at this rate?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

AVERAGE AND PROBABILITY.

82. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average area of the quadrilateral formed by joining the extremities of two chords perpendicular to each other and passing through a point at a distance a from the center of a circle radius R.

83. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the average area of all ellipses whose semi-axis major is a.

84. Proposed by L. C. WALKER, Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

From a point in the circumference of a circle two chords are drawn; find (1) the average radius, and (2) the average area of the circle which touches the two chords and the given circle.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

EDITORIALS.

Mr. L. C. Walker has been elected Associate Professor of Mathematics in Leland Stanford Jr. University, Palo Alto, Cal.

Mr. Edwin Haviland, B. Sc., Swathmore College, 1885; A. M., Cornell University, 1899, has been appointed Instructor in Mathematics in Swathmore College.

John A. Miller, Professor of Astronomy and Mechanics in the Indiana University, received the degree of Doctor of Philosophy at the Summer Convocation of the University of Chicago.

John B. Faught, Associate Professor of Mathematics in Indiana University, who took a course of mathematics at the University of Pennsylvania last year, received the degree of Doctor of Philosophy from that institution at its last commencement.

BOOKS AND PERIODICALS.

Colorado College Studies. Vol. VII. Papers read before the Colorado College Scientific Society. Pamphlet, 48 pages.

The contents of this volume are, "Literature for Children," by Prof. E. S. Parsons; "Warming Up," by Dr. E. G. Lancaster; and "Equations of Motion of a Viscous Liquid," by Mr. P. E. Doudna. The last paper has an historical introduction of much value, beginning with the time of Archimedes and coming down to the time of Stokes and Basset. Section I of the paper treats of the Equations of Motion of a Perfect Fluid, and Section II of the Equations of Motion of a Viscous Fluid. The paper is to be continued in Vol. VIII.
B. F. F.

Plane and Solid Geometry. By William J. Milne, Ph. D., LL. D., President of New York State Normal College, Albany, New York. 12mo. Half Leather, 384 pages. Price, $1.25. Chicago: American Book Co.

A very commendable feature of this new book is the introduction of each theorem by questions so designed as to lead the student to obtain clearly and fully the geometrical concepts of theorems before attempting a demonstration. The book contains 1187 original exercises, giving the teacher a larger number than usual from which to select for class use. The work is one we have no hesitancy in recommending. B. F. F.

Discourse on the Method of Rightly Conducting the Reason and Seeking the Truth in Science. By René Descartes. Translated from the French and Collated with the Latin by John Veitch, LL. D., Late Professor of Logic and Rhetoric in the University of Glasgow. 8vo. Paper Back, 88 pages. Price, 50 cents. Chicago: The Open Court Publishing Co.

While much of the philosophy of the seventeenth century has been destroyed by the relentless investigating powers of the nineteenth century, yet much of what Descartes says in his *Discourse on Method* is as true to-day as when it was written. B. F. F.

The Teaching of Mathematics in High Schools. By E. S. Loomis, Ph. D., Teacher of Mathematics, West High School, Cleveland, Ohio. Pamphlet, 12 pages.

In this paper, which was read before the Ohio State Teachers' Association, at Put-in-Bay, and published in *Education*, Boston, Dr. Loomis has handled his subject admirably. This scholarly address is worthy of a careful study, and the general reading of it will have a very helpful effect on the teaching of mathematics in our secondary schools. Dr. Loomis's long experience as a teacher of mathematics in public schools and colleges enables him to speak authoritatively on the subject. B. F. F.

New Plane and Solid Geometry. A text-book for High Schools, Academies, Normal Schools, and Colleges. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Principal of the State Normal School at Brockport, New York. 8vo. Half Leather, 382 pages. Price, $1.25. Boston: Ginn & Co.

In this new edition the authors have improved very largely on what, to my mind, is the best geometry for the use of schools and colleges that has yet appeared. The style of type has been changed and the whole appearance of the book greatly improved. Many valuable additions in the way of exercises and more complete explanations have been made. The work combines modern scholarship with the rigorous logic of the Euclidean system, thus happily combining vitality and strength in a way to make Geometry the most attractive of the elementary branches of knowledge. B. F. F.

Elementarg Illustrations of the Differential and Integral Calculus. By Augustus De Morgan. New Edition. 8vo. Red Cloth, 144 pages. Price, $1.00 net. Chicago: The Open Court Publishing Co.

In bringing out these reprints, The Open Court Publishing Company is disseminating among the public at large sound views of science and of an adequate and a correct appreciation of the methods by which truth generally is reached. The editor, in the editor's preface of this book, says, "Of these methods, mathematics, by its simplicity, has always formed the type and ideal, and it is nothing less than imperative that its ways of procedure, both in the discovery of new truth and in the demonstration of the necessity and universality of old truth, should be laid at the foundation of every philosophical education. The greatest achievements in the history of thought—Plato, Descartes, Kant—are associated with the recognition of this principle. But it is precisely mathematics, and the pure sciences generally, from which the general educated public and independent students have been debarred, and into which they only rarely attained more than a very meagre insight. The reason of this is twofold. In the first place, the ascendent and consecutive character of mathematical knowledge renders its results absolutely unsusceptible of presentation to persons who are unacquainted with what has gone on before, and so necessitates on the part of its development a thorough and patient exploration of the field from the very beginning, as distinguished from those sciences which may, so to speak, be begun at the end, and which are consequently cultivated with greatest zeal. The second reason is that, partly through the exigencies of academic instruction, but mainly through the martinet traditions of antiquity and the influence of mediaeval logic-mongers, the great bulk of the elementary text-books of mathematics have unconsciously assumed a very repellant form—something similar to what is termed in the theory of protective mimicry in biology the terrifying form. And it is mainly to this formidableness and touch-me-not character of exterior, concealing with harmless body, that the undue neglect of typical mathematical studies are to be attributed." It is gratifying to note that in this country, more attention is given to the study of mathematics. Empty verbosity is giving place to careful and accurate thought and expressions, and true students are finding that the subject most helpful in cultivating correct habits of thought is mathematics. Mathematics because of its varied application, in all departments of art and science, has long been recognized as absolutely essential to the prosecution of a thorough scientific course. But because of the great disciplinary value derived from the study of mathematics, the true classical student is more than ever before availing himself of the benefits to be derived from mathematics by electing it at least through the calculus. B. F. F.

The Mathematical Gazette is issued in February, June, and October, and is edited by W. A. Greenstreet, M. A., of London.

The number for June, in addition to the large number of problems and solutions, contains a "Note on the Sphero-Conic," by Frank Morley; "On the Expression Motion at an Instant," by S. A. Saunder; and "Prismatic Equations," by R. F. Davis. B. F. F.

Annals of Mathematics. A Bi-Monthly Magazine published under the Auspices of the University of Virginia. Edited by William H. Echols. Price, $2.00 per year, in advance.

The June (1899) number, which completes the twelfth volume of this journal, and which is the last to be published at the University of Virginia, contains the following articles: "A Theorem in Determinants," by Dr. E. O. Lovett; "On the Expansion of an Arbitrary Function in Terms of Laplace's Functions," by Prof. W. H. Echols; "On the Relations between Cauchy's Numbers and Bessel's Functions, by Dr. A. Chessin; "On Circuit Integration Over a Straight Line," by Prof. W. H. Echols; "Note on the Invariant Differential Equation," by Prof. J. M. Page. The publication of the *Annals of Mathematics*, so ably conducted at the University of Virginia, will be continued from Harvard University. The first number under the new management appears in October. B. F. F.

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SOME ELEMENTS OF SUBSTITUTION GROUPS.

By DR. G. A. MILLER, Cornell University.

INTRODUCTORY NOTE. It is very gratifying to notice that the number of young men who desire to study modern mathematics is rapidly increasing. The best way to get a start in this direction is to attend some university where there are mathematicians who keep up with the rapid progress that is being made in different lines. While the summer sessions at the universities bring this method within the reach of a large class yet there are many who feel unable to pursue this course. These are frequently discouraged by the difficulties which they meet when they attempt to study the available treatises. The effort which this Journal is making to remove some of these difficulties seems to me to be very commendable. At the request of the editor I shall attempt to state some of the elements of substitution groups in a very simple manner, since it appears that I proceeded somewhat too rapidly at certain places in my articles published in this Journal several years ago.

DEFINITIONS AND NOTATION.

1. The six possible permutations of the three letters a, b, c are the following:

$$\begin{array}{ccc} a & b & c \\ a & c & b \\ b & a & c \\ b & c & a \\ c & a & b \\ c & b & a \end{array}$$

All of these may be obtained from any one of them, by replacing certain letters by others: *e. g.* the second is obtained from the first by replacing b by c and c by b. This operation is called a *substitution*, and it is denoted by bc; hence the substitutions are operations, while the permutations are results. The substitutions by means of which the given permutations are obtained from the

first are, in order, *bc*, *ab*, *abc*, *acb*, *ac*. If these substitutions are applied to *any* one of the given six permutations all the others are obtained. It is convenient to say that a permutation is obtained from itself by the substitution 1 or identity. If we add this substitution to the preceding set we may say that each of the given permutations may be obtained from any one of them by means of one of the following six substitutions :

$$1,\ abc,\ acb,\ ab,\ ac,\ bc.$$

If we apply a given substitution (s) to a given permutation and then apply the same substitution to the resulting permutation we obtain the same result as we would have obtained by applying some other substitution to the original permutation. This substitution is called the square of the given substitution and it is denoted by s^2. *E. g.* if $s=bc$ then $s^2=1$; for if we apply *bc* twice we obtain the original permutation; and if $s=abc$* then $s^2=acb$, etc. In general, if we apply a substitution (s) n times in succession the result is the same as it would have been if we had applied a certain substitution s^n a single time. The smallest positive value of n that satisfies the relation $s^n=1$ is said to be the *order* of s. Hence we say that three (*bc*, *ab*, *ac*) of the given six substitutions are of order two, two (*abc*, *acb*) are of order three, while identity may be said to be of order 0.

If we apply two different substitutions successively, we obtain the same permutation as we would have obtained by applying some substitution a single time. This single substitution which is equivalent to the two substitutions applied successively is said to be their *product*. *E. g.* if we first apply *ab* and then *ac* we obtain the same result as if we had applied *abc*. Hence we say that $ab.ac=abc$. It may be observed that $ac.ab=acb$; *i. e.* the product of two substitutions need not be independent of their order, or the multiplication of substitutions is not always commutative.

Since it is very important that the reader should be able to multiply rapidly and accurately, each of the following products should be verified by the beginner.

$abc.acb=1$	$acb.abc=1$	$ab\ abc=ac$	$ac.abc=bc$	$bc.abc=ab$
$abc.abc=acb$	$acb.acb=abc$	$ab\ acb=bc$	$ac.acb=ab$	$bc.acb=ac$
$abc.ab=bc$	$acb.ab=ac$	$ab.ab=1$	$ac.ab=acb$	$bc.ab=abc$
$abc.ac=ab$	$acb.ac=bc$	$ab.ac=abc$	$ac.ac=1$	$bc.ac=acb$
$abc.bc=ac$	$acb\ bc=ab$	$ab.bc=acb$	$ac.bc=abc$	$bc.bc=1$

When a set of g different substitutions contains all the substitutions which may be obtained by multiplying any two of them or by squaring any one of them

* In performing the operation *abc.abc*, *i. e.* in multiplying *abc* into *abc*, we may say: In the first substitution *a* is replaced by *b* and in the second *b* is replaced by *c*, therefore *a* is replaced by *c* in the product; in the first substitution *c* is replaced by *a* and in the second *a* is replaced by *b*, therefore *c* is replaced by *b* in the product; in the first substitution *b* is replaced by *c* and in the second *c* is replaced by *a*, therefore *b* is replaced by *a* in the product. Since *a* is the first letter this completes the cycle and the given product is *acb*. It is to be observed that we take the letters in order; *e. g.* after finding that *a* is replaced by *c* in the product we next inquired by what *c* is replaced in the first substitution.

the set is called a *substitution group* of order g. The number of different elements or letters that occur in all of the substitutions of the group is said to be the *degree* of the group. Hence the given six substitutions constitute a substitution group of order 6 and of degree 3. When a group is contained in a larger group it is said to be a *subgroup* of the larger group; *e. g.* the group of order 3 and degree 3 whose substitutions are 1, *abc*, *acb* is a subgroup of the given group of order 6. This group of order 6 contains four other subgroups, three of order 2, and one of order 1. It may be observed that the subgroup, identity, occurs in every group.

The substitution *abc* means that a is replaced by b, and then b by c, and finally c by a. If we suppose that these three letters are placed in the given order on the circumference of a circle at intervals of 120° the given substitution is equivalent to a positive rotation of this circle through 120° degrees. Hence such a substitution is called a cycle or a circular substitution. It is evident that the given notation is not unique, for $abc=bca=cab$. In general, if a circular substitution contains n elements it may be written in n ways. This indefiniteness is generally avoided by beginning with the first letter of the alphabet that occurs in the substitution. If this is done the notation becomes unique.

Any substitution whatever is the operation by means of which we may derive a particular permutation from a given arrangement of the elements involved in the substitution, and every rearrangement of the elements of a given permutation leads to a substitution in those elements. Hence we observe that any substitution consists either of a single cycle or of a series of cycles such that no two of them have a common element: *e. g.* the permutation *abcdefghi* is obtained from the permutation *cabedifgh* by means of the substitution *abc.de.fghi*, the periods being used to separate the complete cycles. Since the operations indicated by the different cycles may be performed independently of each other and in any order the given periods may also be interpreted as indicating multiplication.

According to the given notation *any series of letters or elements may be regarded as a substitution provided no letter occurs more than once in the series, and these letters are either not separated by any marks or they are divided into sets of two or more by means of periods.*

Hence there are two methods by means of which we can obtain all the possible substitutions that can be formed with a given number (n) of letters. By the first method we write down all the possible permutations of these n letters and find the substitutions by means of which we can obtain all the $n!$ permutations from any given one of them. By the second method we write down all the possible different substitutions that actually involve the n letters, then those that involve any combination of $n-1$, $n-2$, ... 3, 2 of them. The sum of these will, of course, be $n!$

The beginner would do well if he would find the twenty-four possible substitutions whose degree does not exceed four by means of each of these two methods. It may be remarked that there are just $(n-1)!$ circular substitutions that contain n given letters.

INTEGRATION BY ELLIPTIC INTEGRALS.

By GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

We will first expand the general expression

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}.$$

Let $2\cos\varphi=s+s^{-1}$. $\therefore (1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}=(1+e^2-es-es^{-1})^{-\frac{1}{2}(2m+1)}$

$$=(1-es)^{-\frac{1}{2}(2m+1)}(1-es^{-1})^{-\frac{1}{2}(2m+1)}=\left(1+\frac{2m+1}{2}.es+\frac{(2m+1)(2m+3)}{2^2.\,2\,!}.e^2s^2\right.$$

$$\left.+\frac{(2m+1)(2m+3)(2m+5)}{2^3.\,3\,!}.e^3s^3+\frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4.4\,!}.e^4s^4+\ldots\ldots\right)$$

$$\left(1+\frac{2m+1}{2}.es^{-1}+\frac{(2m+1)(2m+3)}{2^2.\,2\,!}e^2s^{-2}+\frac{(2m+1)(2m+3)(2m+5)}{2^3.\,3\,!}e^3s^{-3}\right.$$

$$\left.+\frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4.4\,!}.e^4s^{-4}+\ldots\ldots\right)=\left[1+\left(\frac{2m+1}{2}.e\right)^2\right.$$

$$\left.+\left(\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{e^2}{2\,!}\right)^2+\left(\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{2m+5}{2}.\frac{e^3}{3\,!}\right)^2+\ldots\right]$$

$$+2\left[\frac{2m+1}{2}.e+\left(\frac{2m+1}{2}\right)^2.\frac{2m+3}{2}.\frac{e^3}{2\,!}+\left(\frac{2m+1}{2}.\frac{2m+3}{2}\right)^2.\frac{2m+5}{2}.\frac{e^5}{2\,!\,.\,3\,!}\right.$$

$$\left.+\ldots\right]\left[\frac{s+s^{-1}}{2}\right]+2\left[\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{e^2}{2\,!}+\left(\frac{2m+1}{2}\right)^2.\frac{2m+3}{2}.\frac{2m+5}{2}.\frac{e^4}{3\,!}\right.$$

$$\left.+\left(\frac{2m+1}{2}.\frac{2m+3}{2}\right)^2.\frac{2m+5}{2}.\frac{2m+7}{2}.\frac{e^6}{2\,!\,.\,4\,!}+\ldots\right]\left[\frac{e^2+e^{-2}}{2}\right]+\ldots.(A).$$

$$\therefore (1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}=\tfrac{1}{2}P_0+P_1\cos\varphi+P_2\cos2\varphi+P_3\cos3\varphi+\ldots$$
$$+P_n\cos n\varphi.$$

When $m=0$, 1, 2, 3, etc., we get

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\tfrac{1}{2}A_0+A_1\cos\varphi+A_2\cos2\varphi+A_3\cos3\varphi+\ldots\ldots\ \ldots\ldots\ldots(B).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{3}{2}}=\tfrac{1}{2}B_0+B_1\cos\varphi+B_2\cos2\varphi+B_3\cos3\varphi+\ \ldots\ldots\ldots\ldots\ldots\ldots(C).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{5}{2}}=\tfrac{1}{2}C_0+C_1\cos\varphi+C_2\cos2\varphi+C_3\cos3\varphi+ \dots\dots\dots\dots (D).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{7}{2}}=\tfrac{1}{2}D_0+D_1\cos\varphi+D_2\cos2\varphi+D_3\cos3\varphi+ \dots\dots\dots\dots (E).$$

....

Let $\sin(\theta-\varphi)=e\sin\theta$, then $\tan\theta=\sin\theta/(\cos\varphi-e)$.

$$\therefore\ \cos(\theta-\varphi)\left(1-\frac{d\varphi}{d\theta}\right)=e\cos\theta.$$

$$\therefore\ \frac{d\varphi}{d\theta}=\frac{\cos(\theta-\varphi)-e\cos\theta}{\cos(\theta-\varphi)}=\frac{\sin\theta\sin\varphi+\cos\theta(\cos\varphi-e)}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore\ \frac{d\varphi}{d\theta}=\frac{(\cos^2\theta\operatorname{cosec}\theta+\sin\theta)\sin\varphi}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{\sin\varphi\operatorname{cosec}\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\sqrt{\frac{(\cos\varphi-e)^2+\sin^2\varphi}{1-e^2\sin^2\theta}}=\sqrt{\frac{1+e^2-2e\cos\varphi}{1-e^2\sin^2\theta}}\ \dots\dots\dots (1_0).$$

Also $\sqrt{(1+e^2-2e\cos\varphi)}=\sqrt{(1-e^2\sin^2\theta)}-e\cos\theta.$

$$\therefore\ \cos\varphi=e\sin^2\theta+\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ \dots\dots\dots\dots\dots\dots (2_0).$$

$$\cos2\varphi=4e^2\sin^4\theta+4e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+1-2(1+e^2)\sin^2\theta\ .\ \dots\dots\dots (3_0).$$

$$\cos3\varphi=4e^3\cos^6\theta+12e^2\sin^4\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+12e\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)$$
$$+4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}-3e\sin^2\theta-3\cos\theta\sqrt{(1-e^2\sin^2\theta)}\dots\dots\dots\dots (4_0).$$

$$\cos4\varphi=1-8e^2\sin^4\theta-16e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}-8\cos^2\theta(1-e^2\sin^2\theta)$$
$$+8e^4\sin^8\theta+32e^3\sin^6\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+48e^2\sin^4\cos^2\theta(1-e^2\sin^2\theta)$$
$$+8\cos^4\theta(1-e^2\sin^2\theta)^2+32e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}\ \dots\dots\dots\dots (5_0).$$

$$\cos5\varphi=16e^5\sin^{10}\theta+80e^4\sin^8\cos\theta\sqrt{(1-e^2\sin^2\theta)}+160e^3\sin^6\theta\cos^2\theta(1-e^2\sin^2\theta)$$
$$+160e^2\sin^4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}+80e\sin^2\theta\cos^4\theta(1-e^2\sin^2\theta)^2$$
$$+16\cos^5\theta(1-e^2\sin^2\theta)^{\frac{5}{2}}-20e^3\sin^6\theta-60e^2\sin^4\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}$$
$$-60\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)-20\cos^3\theta(1-s^2\sin^2\theta)^{\frac{3}{2}}$$
$$+5e\sin^2\theta+5\cos\theta\sqrt{(1-e^2\sin^2\theta)}\dots\dots\dots\dots\dots\dots (6_0),$$

$$\cos 6\varphi = 32e^6\sin^{12}\theta + 192e^5\sin^{10}\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)} + 480e^4\sin^8\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+640e^3\sin^6\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}} + 480e^2\sin^4\theta\cos^4\theta(1-e^2\sin^2\theta)^2$$

$$+192e\sin^2\theta\cos^5\theta(1-e^2\sin^2\theta)^{\frac{5}{2}} + 32\cos^6\theta(1-e^2\sin^2\theta)^3 - 48e^4\sin^8\theta$$

$$-192e^3\sin^6\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)} - 288e^2\sin^4\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$-192e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}} - 48\cos^4\theta(1-e^2\sin^2\theta)^2 + 18e^2\sin^4\theta$$

$$+18\cos^2\theta(1-e^2\sin^2\theta) + 36e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)} - 1 \quad \ldots\ldots\ldots\ldots(7_0).$$

Writing (B) in the following form :

$$(1+e^2-es-es^{-1})^{-\frac{1}{2}} = \tfrac{1}{2}A_0 + \tfrac{1}{2}A_1(s+s)^{-1}) + \ldots\ldots \; + \tfrac{1}{2}A_n(s^n+s^{n-1}) + \ldots\ldots\ldots(8_0).$$

Differentiating (8_0) we get,

$$e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{3}{2}} = A_1(1-s^{-2}) + 2A_2(s-s^{-3})$$

$$+3A_3(s^2-s^{-4}) + \ldots\ldots + nA_n(s^{n-1}-s^{-(n+1)}) + \ldots.(9_0).$$

From (8_0) and (9_0) we get,

$$\tfrac{1}{2}e(1-s^{-2})[A_0 + A_1(s+s^{-1}) + A_2(s^2+s^{-2}) + \ldots\ldots + A_n(s^n+s^{-n}) + \ldots\ldots]$$

$$=(1+e^2-es-es^{-1})[A_1(1-s^{-2}) + 2A_2(s-s^{-3}) + 3A_3(s^2-s^{-4})$$

$$+ \ldots\ldots + nA_n(s^{n-1}-s^{-(n+1)}) + \ldots\ldots].$$

Equating coefficients of s^n we get,

$$\tfrac{1}{2}e(A_n - A_{n+2}) = (n+1)(1+e^2)A_{n+1} - e[nA_n + (n+2)A_{n+2}].$$

$$\therefore \; A_{n+2} = \frac{2(n+1)}{2n+3} \cdot \frac{1+e^2}{e} A_{n+1} - \frac{2n+1}{2n+3} A_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(10_0).$$

$\therefore$ When A_n and A_{n+1} are known we can easily find A_{n+2}.
Multiplying (B) by $\cos\varphi$ we get,

$$\frac{\cos\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = \tfrac{1}{2}A_0\cos\varphi + \tfrac{1}{2}A_1(1+\cos 2\varphi) + \tfrac{1}{2}A_2(\cos\varphi + \cos 3\varphi) + \ldots.(11_0).$$

$$\text{Also} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = F(e,\ \tfrac{1}{2}\pi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,d\theta = E(e,\ \tfrac{1}{2}\pi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Integrating both sides of (B) and (11_0) between the limits 2π and 0, we get with the aid of (1_0), (2_0), (1), (2), the following:

$$\pi A_0 = \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = 4F(e,\ \tfrac{1}{2}\pi).$$

$$\therefore\ A_0 = (4/\pi)F(e,\ \tfrac{1}{2}\pi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12_0).$$

$$\pi A_1 = \int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4e\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{1-e^2\sin^2\theta}} + \int_0^{2\pi} \cos\theta d\theta$$

$$= \frac{4}{e}\left[\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} - \int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,d\theta\right] = \frac{4}{e}[F(e,\ \tfrac{1}{2}\pi) - E(e,\ \tfrac{1}{2}\pi)].$$

$$\therefore\ A_1 = (4/\pi e)[F(e,\ \tfrac{1}{2}\pi) - E(e,\ \tfrac{1}{2}\pi)].$$

[To be Continued.]

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION."

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from October Number.]

CHAPTER III.

MULTIPLICATION OF EXTENSIVE QUANTITIES. DIFFERENT KINDS OF MULTIPLICATION.

22. In the multiplication of extensive quantities expressed in terms of *units*, it is assumed that the distributive law holds, and that numerical coefficients may be treated as in elementary algebra (16).

Thus if $a=\Sigma\alpha_r e_r$ and $b=\Sigma\beta_s e_s$ are two extensive quantities in which α and β are numbers and the e's are extensive *units*, we may write

$$ab=[\Sigma\alpha_r e_r.\Sigma\beta_s e_s]=\Sigma\alpha_r\beta_s[e_r e_s],$$

that is to say, in the result each term of the multiplier is multiplied into every term of the multiplicand, and the partial products are added.

Notice that the law is assumed to hold only when the two factors are sums of *units*. In the theorems which follow it is shown that the same law applies when the factors are sums of *quantities*.

23. Before going on to the proofs of these theorems, we will illustrate the question involved by an example. (See Art. 9).

Let

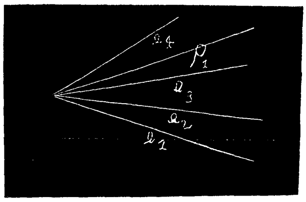
$$\rho_1=r_{11}e_1+r_{12}e_2+r_{13}e_3+\ldots\ldots$$

$$\rho_2=r_{21}e_1+r_{22}e_2+r_{23}e_3+\ldots\ldots$$

Also, let

$$a=\alpha_1\rho_1+\alpha_2\rho_2+\ldots\ldots$$

$$b=\beta_1\rho_1+\beta_2\rho_2+\ldots\ldots$$



It is to be shown, then, that if $\rho_1\rho_2=\Sigma r_{1r}r_{2s}[e_r e_s]$, $ab=\Sigma\alpha_r\beta_s[\rho_r\rho_s]$.

The proof will be based on the definitions laid down and the theorems proved in the last chapter.

Remark.—It should be kept in mind in Articles 24—30 that α, β, ... denote numbers, the e's denote extensive *units* (11), and a, b, ... denote extensive *quantities* (12). Square brackets are used to indicate that the quantities inside are extensive quantities whose product is required.

24. *To show that* $[\Sigma\alpha_r e_r.b]=\Sigma\alpha_r[e_r b]$, *i. e.* to show that in multiplying $\Sigma\alpha_r e_r$ by b each term of $\Sigma\alpha_r e_r$ is multiplied by b.

Proof.—Let $b=\Sigma\beta_s e_s$. Then

$$[\Sigma\alpha_r e_r.b]\equiv[\Sigma\alpha_r e_r.\Sigma\beta_s e_s]=\Sigma\alpha_r\beta_s[e_r e_s] \quad (22)$$

$$=\Sigma\alpha_1\beta_s[e_1 e_s]+\Sigma\alpha_2\beta_s[e_2 e_s]+\ldots\ldots \quad (14)$$

$$=\alpha_1\Sigma\beta_s[e_1 e_s]+\alpha_2\Sigma\beta_s[e_2 e_s]+\ldots\ldots \quad (15)$$

$$=\alpha_1[e_1.\Sigma\beta_se_s]+\alpha_2[e_2.\Sigma\beta_se_s]+\ \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$=\alpha_1[e_1.b]+\alpha_2[e_2.b]+\ldots\ldots=\Sigma\alpha_r[e_rb].$$

25. *To show that* $[(a+b+\ldots\ldots)p]=[ap]+[bp]+\ldots\ldots$

$$[p(a+b+\ldots\ldots)]=[pa]+[pb]+\ldots\ldots$$

Proof.—Let $a=\Sigma\alpha_re_r$, $b=\Sigma\beta_re_r\ldots\ldots$ Then

$$[(a+b+\ldots\ldots)p]=[(\Sigma\alpha_re_r+\Sigma\beta_re_r+\ldots\ldots)p]=[\Sigma(\alpha_r+\beta_r+\ldots\ldots)e_r.p]\ (14)$$

$$=\Sigma(\alpha_r+\beta_r+\ldots\ldots)[e_rp]\ \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

$$=\Sigma[\alpha_re_r.p]+\Sigma[\beta_re_r.p]+\ \ldots\ldots\ldots\ldots\ldots\ldots(14,\ 24)$$

$$[ap]+[bp]+\ldots\ldots$$

26. *To show that* $[(\alpha a)b]=\alpha[ab]$, and $[b(\alpha a)]=\alpha[ba]$.

Proof.—Let $a=\Sigma\alpha_re_r$. Then

$$[(\alpha a)b]\equiv[(\alpha\Sigma\alpha_re_r)b]=[\Sigma\alpha\alpha_re_r)b]\ \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

$$=\Sigma\alpha\alpha_r[e_rb]\ (24)\ =\alpha[\Sigma\alpha_re_r.b]\ (16,\ 24)\ =\alpha[ab].$$

The other formula is obtained by making b the first factor in the above proof.

27. *To show that* $[(\alpha a+\beta b+\ldots\ldots)p]=\alpha[ap]+\beta[bp]+\ldots\ldots$

and $[p(\alpha a+\beta b+\ldots\ldots)]=\alpha[pa]+\beta[pb]+\ldots\ldots$

Proof.—$[(\alpha a+\beta b+\ldots\ldots)p]=[(\alpha a)p]+[(\beta b)p]+\ \ldots\ldots\ldots\ldots(25)$

$$=\alpha[ap]+\beta[bp]+\ \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

28. We are now in position to show that the distributive law holds when the two factors are sums of multiples of extensive *quantities* as well as when they are sums of multiples of units.

Proof.—$[\Sigma\alpha_ra_r.\Sigma\beta_sb_s]=\Sigma\alpha_r[a_r.\Sigma\beta_sb_s]\ \ldots\ldots\ldots\ldots\ldots\ldots(27)$

$$=\Sigma\alpha_r(\Sigma\beta_s[a_rb_s])\ (27)\ =\Sigma\alpha_r\beta_s[a_rb_s]\ \ldots\ldots\ldots(16)$$

This theorem holds also for any number of factors, as can be shown by mathematical induction.

29. Let us denote a product containing a number of factors, a, b, ...

by $P_{a,\ b\ldots\ldots}$ In such a product suppose a factor p equals $qa+rb+\ldots\ldots$ where q, r, $\ldots\ldots$ are numbers; to show that

$$P_{qa+rb+\ldots\ldots}=q.P_a+r.P_{b+\ldots\ldots}$$

PROOF.—However the product may be made up, we can always regard p as combined with another factor, then this product with other factors in turn. In each of the multiplications in which p enters as one factor Art. 27 applies.

30. *To show that* $P_{qa,\ rb,\ sc,\ \ldots\ldots}=qrs\ldots\ldots P_{a,\ b,\ c\ldots\ldots}$

PROOF.—By 29, $P_{qa}=q.P_a$. Then

$$P_{qa,\ rb,\ sc\ldots\ldots}=q.P_{a,\ rb,\ sc}\ \ldots\ldots=qrs\ldots\ldots\ P_{a,\ b,\ c}\ \ldots\ldots$$

It evidently follows that $P_{qa,\ ra}=P_{ra,\ qa}$.

31. *Different Kinds of Multiplication.*—Different kinds of multiplication are obtained by laying down different laws for simplifying a distributed product.

To illustrate:—

(1) $(\alpha_1 e_1+\alpha_2 e_2)(\beta_1 e_1+\beta_2 e_2)=\alpha_1\beta_1 e_1^2+\alpha_1\beta_2 e_1 e_2+\alpha_2\beta_1 e_2 e_1+\alpha_2\beta_2 e_2^2$.

In ordinary algebra the result is simplified by supposing $e_1 e_2=e_2 e_1$. The result in this way becomes

$$\alpha_1\beta_1 e_1^2+(\alpha_1\beta_2+\alpha_2\beta_1)e_1 e_2+\alpha_2\beta_2 e_2^2,$$

or, say, $m_1 e_1^2+m_2 e_1 e_2++m_3 e_2^2$, where the m's are numerical coefficients.

(2) Similarly, writing three factors, we get,

$$(\alpha_1 e_1+\alpha_2 e_2)(\beta_1 e_1+\beta_2 e_2)(\gamma_1 e_1+\gamma_2 e_2)=m_1 e_1^3+m_2 e_1^2 e_2+m_3 e_1 e_2^2+m_4 e_2^3$$

by supposing, as in ordinary algebra, that

$$\begin{cases} e_1 e_1 e_2=e_1 e_2 e_1=e_2 e_1 e_1 \\ e_1 e_2 e_2=e_2 e_1 e_2=e_2 e_2 e_1 \end{cases}$$

Here the assumed law of simplification reduces a product which would otherwise have eight terms to one of four.

(3) The law of simplification in quaternions (which is a branch of mathematics using a certain kind of extensive quantities) may be seen by multiplying together two factors of three terms each.

Thus, $(\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3)(\beta_1 e_1+\beta_2 e_2+\beta_3 e_3)=$

$$-(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)+(\alpha_1\beta_2-\alpha_2\beta_1)e_3+(\alpha_2\beta_3-\alpha_3\beta_2)e_1+(\alpha_3\beta_1-\alpha_1\beta_3)e_2,$$

by supposing $e_1^2=e_2^2=e_3^2=-1$, $e_1 e_2=e_3$, $e_2 e_3=e_1$, and $e_3 e_1=e_2$.

32. Definition.—A multiplication is said to be *linear* when the same laws of simplication of the distributed product continue to hold when numerically derived quantities (10) replace the given units.

33. *To show that there are but four kinds of linear multiplication.*

Let

$$(a) \qquad \Sigma\alpha_{rs}[e_re_s]=0$$

express a simplification law in the product $\Sigma\alpha_r\beta_s[e_re_s]$.

Let, now, e_r be replaced by $\Sigma x_{r,u}\,e_u$ and e_s by $\Sigma x_{sv}e_v$. We thus get

$$\Sigma\alpha_{rs}[\Sigma x_{ru}e_u.\Sigma x_{sv}e_v]=0,$$

$$\text{whence, } \Sigma\alpha_{rs}\Sigma x_{ru}x_{sv}[e_ue_v]=0, \text{ (28)},$$

$$\text{or, } \Sigma\alpha_{rs}x_{ru}x_{sv}[e_ue_v]=0. \text{ (16)}$$

This equation is symmetrical in r and s and u and v and evidently will continue to hold true when these letters are interchanged. This gives

$$\Sigma\alpha_{sr}x_{sv}x_{ru}[e_ve_u]=0.$$

Adding the last two equations, we have

$$(b) \qquad \Sigma x_{ru}x_{sv}\{\alpha_{rs}[e_ue_v]+\alpha_{sr}[e_ve_u]\}=0.$$

Equation (b) may also be gotten by multiplying $\Sigma\alpha_{rs}x_{ru}x_{sv}[e_ue_v]$ by 2 and arranging the result with reference to equal coefficients, $(x_{rv}x_{sv})$. This derivation shows (b) to be a necessary, and not, as might appear, an arbitrary inference from the given equation.

Now from the nature of the case the coefficients x_{ru}, x_{sv} must be capable of having any values, as would the x's in Articles 6—9. If we assume that the *products* $x_{ru}x_{sv}$ are arbitrary, then from the theory of equations we have

$$(c) \qquad \alpha_{rs}[e_ue_v]+\alpha_{sr}[e_ve_u]=0,$$

true for all values of r and s and u and v.*

If we put $u=v$ in (c) we get

$$(d) \qquad (\alpha_{rs}+\alpha_{sr})[e_ue_u]=0.$$

This equation is satisfied either by assuming (1) $\alpha_{rs}+\alpha_{sr}=0$, *i. e.*, $\alpha_{rs}=-\alpha_{sr}$; or, by assuming (2), $[e_ue_v]=0$.

*Grassmann's derivation of (c) does not assume that the products $x_{ru}x_{sv}$ are arbitrary. The writer gave another demonstration which does not assume this before the Mathematical Section of the American Association for the Advancement of Science, 1899 meeting. Though somewhat simpler than Grassmann's proof, it would add materially to the length of this article.

(1) If $\alpha_{rs}=-\alpha_{sr}$, and we make this substitution in (c), there results

$$\alpha_{rs}\{[e_u e_v]-[e_v e_u]\}=0.$$

In this equation either $\alpha_{rs}=0$, or $[e_u e_v]=[e_v e_u]$. If $\alpha_{rs}=0$, all the coefficients reduce to zero, and equation (a) vanishes identically, which is contrary to hypothesis. If

$$(e) \qquad [e_u e_v]-[e_v e_u]=0, \text{ or } [e_u e_v]=[e_v e_u],$$

we have the law for a form of multiplication of extensive quantities which is analogous to ordinary multiplication in algebra. See Art. 31, (1).

(2) If we say $[e_u e_u]=0$, it is equivalent to making in equation (a) $\alpha_{rr}=1$, and all the other coefficients equal to 0. Making this substitution in (c), we get

$$(f) \qquad [e_u e_v]+[e_v e_u]=0,$$

which implies $[e_u e_u]=0$, as may be seen by making $u=v$ in (f).

We have seen that equations (e) and (f) are necessary conditions in order that a multiplication may be linear. That they are sufficient conditions may be seen as follows: If we start with

$$(a) \qquad [e_r e_s]\pm[e_s e_r]=0,$$

and substitute as above we get

$$(c) \qquad [e_r e_s]\pm[e_s e_r]=0.$$

Hence, by definition, (32) (e) and (f) give linear multiplications.

We have then four kinds of linear multiplication, viz :

1st. That in which there are no simplifying equations.

2nd. That in which all the coefficients in (a) are identically 0.

3d. That whose law of simplification is $[e_u e_v]=[e_v e_u]$:

4th. That whose law of simplification is $[e_u e_v]=-[e_v e_u]$.

As between (e) and (f), the latter gives the simpler species of multiplication. To see this let us take the distributed product in (1) Art. 31. Equation (e) reduces the product, as we saw in that Article to three terms. But taking (f) as the simplification law, we get a single term, viz., $(\alpha_1\beta_2-\alpha_2\beta_1)e_1e_2$.

34. The *Ausdehnungslehre* concerns itself very largely with the operation of multiplication, especially with what is called *combination* multiplication. This multiplication is based on the law described in the last Article, viz., $e_r e_s=-e_s e_r$, which also implies $e_r\, e_r=0$.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\frac{.297}{1.003}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain %?

II. Solution by JOHN M. HOWIE, Professor of Mathematics, The Nebraska State Normal, Peru, Neb.

In the solution of the above in the August-September MONTHLY by Professors Schmitt and Zerr, the following statements are made:

$$.\dot{2}9\dot{7}=\frac{297}{999}=\frac{11}{37}\,;\ 1.00\dot{3}=1\frac{1}{300}=\frac{301}{300}.\quad \frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}\,;\ .\frac{3300}{11137}=\frac{330}{11137}.$$

It seems to me the statements are inconsistent.

$$\text{If } .\frac{3300}{11137}=\frac{300}{11137} \text{ then also } .\dot{2}9\dot{7}=.\frac{297}{999}=.\frac{11}{37}=\frac{11}{370}.$$

But if $.\dot{2}9\dot{7}=\frac{297}{999}=\frac{11}{37}$, then $.\frac{3300}{11137}=\frac{3300}{11137}$.

The latter I think to be correct.

The following solution seems to me to be correct:

$$.\dot{2}9\dot{7}=.\frac{297}{999}=.\frac{11}{37}=\frac{11}{37}\,;\ 1.00\dot{3}=1\frac{1}{300}=\frac{301}{300}.\quad \frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}\,;\ .\frac{3300}{11137}=\frac{3300}{11137}.$$

$$\therefore\ \$4.\frac{.\dot{2}9\dot{7}}{1.00\dot{3}}=\$4\frac{3300}{11137}=\$\frac{47848}{11137}=\text{cost price.}$$

$$\$6.\frac{1000}{33337}=\$6\frac{1000}{33337}=\$\frac{201022}{33337}=\text{selling price.}$$

$$\therefore\ \$\frac{201022}{33337}-\$\frac{47848}{11137}=\$\frac{17396574}{100344437}=\text{gain.}$$

$$\frac{17396574}{100344437}\div\frac{47848}{11137}=.40\frac{1901935}{5388881}=40\frac{1901935}{5388881}\%=\text{gain per cent.}$$

117. Proposed by MARCUS BAKER, U.S. Coast and Geodetic Survey, 1905 Sixteenth St., Washington, D.C.

A landed man two daughters had,
  And both were very fair;
He gave to each a piece of land,
  One round, the other square.

At twenty shillings an acre, just,
  Each piece its value had;
The shillings that did compass each,
  For it exactly paid.

If 'cross a shilling be an inch,
(As it is, very near),
Which had the larger portion, she
That had the round or square?

Also, how many acres did each receive ?
[Does anyone know the history of this problem ?]

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and ELMER SCHUYLER, Reading, Pa.

Let r=radius of circle, x=side of square, both in rods.

In this solution we regard the shillings on the perimeter of the circle as tangent at the extremities of their diameters.

$\therefore \dfrac{20\pi r^2}{160}=2\pi r\times 16\frac{1}{2}\times 12.$

$\therefore r=3068$ rods ; $\dfrac{\pi r^2}{160}=184816.87224$ acres.

$\dfrac{20x^2}{160}=4x\times 16\frac{1}{2}\times 12.$

$\therefore x=6336$ rods ; $\dfrac{x^2}{160}=250905.6$ acres.

II. Solution by D. G. DORRANCE, Jr., Camden, N. Y.

(1) Let $2\pi R=x$=number of inches in circumference of circular parcel, and (2) $\pi R^2=x/20$ acres$=313632x$ square inches.

From (1), $R=\dfrac{x}{2\pi}$, or $R^2=\dfrac{x^2}{4\pi^2}$.

From (2), $R^2=\dfrac{313632x}{\pi}$. Then $\dfrac{x^2}{4\pi^2}=\dfrac{313632x}{\pi}$.

Whence $x=313632(4\pi)=3941225.1648$ inches in circumference.

$x/20=197061.25824$, number of acres in circular parcel.

(1) Let $4y=x$=number of inches around the square.

(2) Let $y^2=x/20$ acres$=313632x$ square inches.

From (1), $y=x/4$, or $y^2=x^2/16$.

From (2), $y^2=313532x$.

Then $x^2/16=313632x$.

Whence $x=5018112$. $x/20=250905.6$=number of acres in square parcel.

118. Proposed by J. F. TRAVIS, Student in Ohio State University, Columbus, Ohio.

The present worth of a note due January 1, 1896, was $74,200, when discounted at 4% true discount. The present worth of another note, due July 1, 1896, whose face value was the same as that of the first note, was $68,900 when discounted at 8% true discount. Find the face of the notes and the date when given, supposing the second note to have been given the same day the first note was. Solve by arithmetic.

I Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let F=face of each note, t=number of years before January 1, 1896, the notes were given.

Then $(1+.04t)(74200)=[(1+.08(t+\frac{1}{2})](68900)$.

$\therefore$ $68900\times .08t=5300(1+.04t)$.

$\therefore$ $t=1$ year. $\therefore$ time was January 1, 1895.

$F=(74200)(1.04)=\$77168$.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Ler x=the number of days the first note had to run, and $x+180$ the num-the second had to run ; regarding 360 days to the year.

Then we have, since the face of each note was the same,

$$74200\left(1+\frac{4}{100}\frac{x}{360}\right)=68900\left(1+\frac{8}{100}\frac{x+180}{360}\right).$$

Whence, dividing by 100 and transposing,

$$53=\frac{5512x+992160-2968x}{36000}$$

or, $2544x=53(36000)-992160=915840$.

$x=360$ days, or date of each note was January 1st, 1895.

The face value must have been \$74200+the discount, which is 4% of \$74200 or \$2968 ; that is \$77168.

119. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$1\,\frac{9}{.100}$. The selling price is \$1,000. What is the gain per cent.?

Solution by C. C. BEBOUT, Professor of Mathematics, High School, Elgin, Ill., and J. M. HOWIE, Professor of Mathematics, The Nebraska State Normal School, Peru, Neb.

I do not think that the solution of Arithmetic problem 112, published in the August-September MONTHLY is correct. My solution of No. 119 will show wherein I differ. C. C. BEBOUT.

$\$1.\frac{9}{.001}=\$1.+\$\frac{9}{.001}=\$1.+\$9000=\$9001$, the selling price.

If the article costs \$9001., and sells for \$1000, there is no gain, but a loss of \$8001, which is 88.89+% of \$9001.

The question seems to be as to the meaning of the decimal point following the 1 in the expression $1.\frac{9}{.001}$. A common fraction can not have place value, nor can it give place value as a digit in a number. It is simply added to the number to which it is attached and is a fraction of the unit to which it is attached. If it stands alone it is a fraction of the understood (or named) unit. We would not write $3\frac{1}{2}4$ for 354, nor $2\frac{1}{2}$. for 25. Also $4\frac{1}{2}=4+\frac{1}{2}=4.+\frac{1}{2}=4.\frac{1}{2}$. Every in-

teger in our decimal notation is, in theory, followed by a decimal point, and it can make no difference in the meaning of the expression whether the decimal point is expressed or understood. $4.\frac{1}{2}$ must equal 4.5 and not 4.05, which is the equivalent of $4.0\frac{1}{2}$.

$$\text{So } 1.\frac{9}{.001}=1\frac{9}{.001}=1+\frac{9}{.009}.$$

Also solved by *COOPER D. SCHMITT*, *D. A. LEHMAN*, *ELMER SCHUYLER*, and the *PROPOSER*. These contributors agree that the result is $10\frac{890}{901}\%$. To my mind, the solution and discussion of the problem as published above are correct. Ed. F.

ALGEBRA.

94. Proposed by J. W. YOUNG, Columbus, Ohio.

$$\text{Solve}: \left[\frac{x^2+14x+1}{p^4+14p^2+1}\right]^3=\frac{x(x-1)^4}{p^2(p^2-1)^4}.$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $p^2(p^2-1)^4=A$, $(p^4+14p^2+1)^3=B$.

$$\therefore Ax^6+42Ax^5+591Ax^4+2828Ax^3+591Ax^2+42Ax+A$$
$$=Bx^5-4Bx^4+6Bx^3-4Bx^2+Bx.$$

$$\therefore Ax^6+(42A-B)x^5+(591A+4B)x^4+(2828A-6B)x^3$$
$$+(591A+4B)x^2+(42A-B)x+A=0.$$

$$\therefore A[x^3+(1/x^3)]+(42A-B)[x^2+(1/x^2)]+(591A+4B)[x+(1/x)]$$
$$+(2828A-6B)=0.$$

$$\therefore A[x+(1/x)]^3+(42A-B)[x+(1/x)]^2+(588A+4B)[x+(1/x)]$$
$$+(2744A-4B)=0.$$

Let $[x+(1/x)]=y$.

$$\therefore p^2(p^2-1)^4y^3-(p^{12}+759p^8+2576p^6+759p^4+1)y^2$$
$$+4(p^{12}+189p^{10}+3p^8+3710p^6+3p^4+189p^2+1)y$$
$$-4(p^{12}-644p^{10}+3335p^8-1288p^6+3335p^4-644p^2+1)=0.$$

Let $a=p^2+1/p^2$.

$$\therefore (2-a)^2y^3-(a^3+756a+2576)y^2+4(a^3+189a^2+3332)y$$
$$-4(a^3-644a^2+3332a)=0.$$

$$\therefore (y-a)[(2-a)^2y^2-4(a^2+188a+644)y+4(a^2-644a+3332)]=0.$$

$$\therefore\ y=a \text{ and } y=\frac{2(a^2+188a+644)\pm64\sqrt{(a^3+30a^2+252a+392)}}{(2-a)^2}.$$

Let $p+1/p=b$.

$$\therefore\ y=b^2-2,\ y=\frac{2(b^2+28b+68)}{(b-2)^2},\ y=\frac{2(b^2-28b+68)}{(b+2)^2}.$$

$$\therefore\ p^2x^2-(p^4+1)x+p^2=0.$$

$$(b-2)^2x^2-2(b^2+28b+68)x+(b-2)^2=0.$$

$$(b+2)^2x^2-2(b^2-28b+68)x+(b+2)^2=0.$$

$$x=p^2,\ x=1/p^2,\ x=\frac{(p^4+28p^3+70p^2+28p+1)\pm8(p^3+7p^2+7p+1)\sqrt{p}}{(p-1)^4}.$$

$$x=\frac{(p^4+28p^3+70p^2-28p+1)\pm8(p^3-7p^2+7p-1)\sqrt{(-p)}}{(p+1)^4}.$$

II. Solution by ELMER SCHUYLER, Reading, Pa.

Let $p^2=t$, $x^2+1=xz$, and $t^2+1=tv$. Then

$$\frac{x^3(z+14)^3}{t^3(v+14)^3}=\frac{x^2(z-2)^2}{t^3(v-2)^2}.\quad \therefore\ \left(\frac{z+14}{v+14}\right)^3=\left(\frac{z-2}{v-2}\right)^2.$$

By inspection, $z=v$ fulfills conditions.

$$\therefore\ \frac{x^2+1}{x}=\frac{p^4+1}{p^2},$$ which gives $x=p^2$ or $1/p^2$ (two answers).

Since $$\frac{(z+14)^3-(v+14)^3}{(v+14)^3}=\frac{(z-2)^2-(v-2)^2}{(v-2)^2},$$

we can eliminate, factor $z-v$, and get a quadratic in z, or a quadro-quadratic in x, which equations give

$$x=\left[\frac{1+\theta\sqrt{p}}{1-\theta\sqrt{p}}\right]^4,\ \theta^4=1.$$

[Quadro-quadratic meaning quadratic equation whose unknown quantity is a quadratic.]

III. Solution by E. D. ROE, Jr., A. M., Ph. D., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Put $x=z^2$ and divide numerators by z^6, and denominators by p^6. We get:

$$\left(\frac{z^2+14+(1/z^2)}{p^2+14+(1/p^2)}\right)^3=\frac{[z-(1/z)]^4}{[p-(1/p)]^4}\ldots\ldots(1).$$

We notice that $z=p$, and $1/p$.
Put $z-1/z=y$, $p-1/p=r$, and (1) becomes

$$\left(\frac{y^2+16}{r^2+16}\right)^3=\left(\frac{y}{r}\right)^4 \ldots\ldots(2).$$

Multiplying out, we have,

$$y^6r^4+3.16y^4r^4+3.16^2y^2r^4+16^3r^4-r^6y^4-3.16r^4y^4-3.16^2r^2y^4-16^3y^4=0, \text{ or}$$

$$r^4y^4(y^2-r^2)-3.16^2r^2y^2(y^2-r^2)-16^3(y^2+r^2)(y^2-r^2)=0,$$

$$\text{or } (y^2-r^2)[r^4y^4-16^2(3r^2+16)y^2-16^3r^2]=0 \;\ldots\ldots(3).$$

From this we have $y^2-r^2=0\ldots\ldots(4)$,

$$\text{or } r^4y^4-16^2(3r^2+16)y^2-16^3r^2=0\ldots\ldots(5).$$

By replacing y, and r, (4) gives

$$z^2-2+(1/z^2)=x-2+(1/x)=p^2-2+(1/p^2), \text{ or } x+(1/x)=p^2+(1/p^2),$$

$$\text{whence } x^2-[p^2+(1/p^2)]x+1=0, \text{ and } (x-p^2)[x-(1/p^2)]=0,$$

and $x=p^2$, $x=1/p^2$, as before noticed.
Solving (5) for y^2, we have

$$y^2=\frac{2.8^2(3r^2+16)\pm 8^2\sqrt{[4(3r^2+16)^2+r^6]}}{r^4}=x-2+1/x, \text{ and } x+1/x=$$

$$=\frac{2(p^2-1)^4+2.8^2[3(p^2-1)^2+16p^2]p^2\pm 8^2p\sqrt{4[3(p^2-1)^2+16p^2]p^2+(p^2-1)^6}}{(p^2-1)^4}$$

The expression under the radical sign is found to be the square of $p^6+15p^4+15p^2+1$. Taking the upper sign and dividing out the factor $(p+1)^4$ from numerator and denominator,

$$x+1/x=\frac{2(p^4+28p^3+70p^2+28p+1)}{(1-p)^4}.$$

Taking the lower sign and similarly dividing out the factor $(p-1)^4$,

$$x+1/x=\frac{2(p^4-28p^3+70p^2-28p+1)}{(1+p)^4}.$$

It is seen that both these values of $x+1/x$ are contained in the formula,

$$\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon^2 p)^4},$$

where ε is a fourth root of unity, as only even powers of ε and $\sqrt{p}$ occur, and every even power of ε is $+1$ or -1. For $\varepsilon=+1$, or -1, we get the first value; and for $\varepsilon=+i$, or $-i$, we get the second value of $x+1/x$, and these are the only values which the formula admits. Thus

$$x+1/x=\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon^2 p)^4}=\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon\sqrt{p})^4(1+\varepsilon\sqrt{p})^2}$$

$$=\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4+\left(\frac{1+\varepsilon\sqrt{p}}{1+\varepsilon\sqrt{p}}\right)^4=\lambda+1/\lambda$$

if λ denote $\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4$. Then as before

$$x^2-[\lambda+(1/\lambda)]x+1=(x-\lambda)[x-(1/\lambda)]=0,$$

and $x=\lambda$, or $x=1/\lambda$, but when ε takes its four values, the group of values represented by λ, is the same as the group represented by $1/\lambda$, though not corresponding value by value for the same value of ε. The values of λ and $1/\lambda$ are the same for 1, and -1, -1 and 1, i and $-i$, $-i$ and i, respectively.

The six values of x are

$$x=p^2,\ x=1/p^2, \text{ and either } x=\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4, \text{ or } x=\left(\frac{1-\varepsilon\sqrt{p}}{1+\varepsilon\sqrt{p}}\right)^4$$

for the last four values, or other similar expressions containing ε, which are easily formed.

GEOMETRY.

119. Proposed by WILLIAM HOOVER., A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A sphere touches each of two straight lines which are inclined to each other at a right angle but do not meet; show that the locus of its center is an hyperbolic paraboloid.

I. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If the common perpendicular to the two given lines be taken as the Z-axis, its length represented by $2c$, its middle point be taken as origin, a plane parallel to the two lines as xy-plane, and the projections of the lines on the xy-plane as the X and Y-axis, respectively, the equations of the lines will be $y=0$, $z=c$, and $x=0$, $z=-c$. The square of the distance of the point (x', y', z'), any point in the required locus, from the first line is $y'^2+(c-z')^2$, and from the second

line $x'^2+(c+z')^2$. These distances are equal, and we get $x^2-y^2+4cz=0$ as the required equation. This is the equation of a hyperbolic paraboloid.

A more general problem is to find the locus of points equi-distant from any two non-intersecting straight lines in space.

If the axes are taken as above, except that the bisectors of the angles formed by the projections of the given lines on the xy-plane are taken as the X and Y-axis, the resulting equation will be

$$\frac{mxy}{1+m^2}+cz=0,$$

which is also the equation of a hyperbolic paraboloid.

II. Solution by the PROPOSER.

With coördinate axes rectangular, the two given lines may be taken as

$$y=mx,\ z=c\ldots\ldots(1),\ \text{and}\ y=-mx,\ z=-c\ldots\ldots(2).$$

Let the sphere be $(x-x')^2+(y-y')^2+(z-z')^2=r^2\ldots\ldots$ (3).

(1) intersects (3) where

$$(1+m^2)x^2-2(x'+my')x+(x'^2+y'^2+c^2-2cz'+z'^2-r^2)=0\ldots\ldots\ (4).$$

(1) will then be tangent to (3) if

$$(x'+my')^2=(1+m^2)(x'^2+y'^2+c^2-2cz'+z'^2-r^2)\ldots\ldots(5).$$

Similarly, (2) will be tangent to (3) if

$$(x'-my')^2=(1+m^2)(x'^2+y'^2+c^2+2cz'+z'^2-r^2)\ldots\ldots(6).$$

(5)$-$(6) gives, $mx'y'=-c(1+m^2)z'\ldots\ldots(7)$, the required locus of the center (x', y', z') of (3). But (7) is an hyperbolic paraboloid.

III. Solution by J. W. YOUNG, Graduate Student, Ohio State University, Columbus, Ohio.

Let the two straight lines be

$$\frac{x}{0}=\frac{y}{0}=\frac{z}{1};\quad \frac{x-a}{0}=\frac{y}{1}=\frac{z}{0}.$$

The center of a sphere, which touches these two straight lines will always be equidistant from them. Hence equating distances, putting (x_1, y_1, z_1) for the center, we have

$$x_1^2+y_1^2=(x_1-a)^2+z_1^2.$$

Whence the required locus is easily seen to be

$$y^2-z^2=-2ax+a^2,$$

a hyperbolic paraboloid.

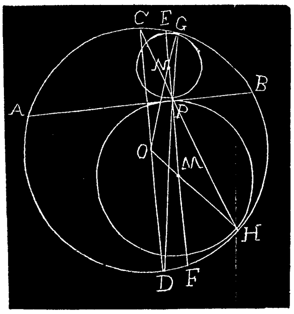
Excellent solutions were received from *G. B. M. ZERR*, and *J. SCHEFFER*.

120. Proposed by P. C. CULLEN, Principal of Public Schools, Indianola, Neb.

Draw a circle tangent to a given circle and tangent to a given chord at a given point.

I. Solution by M. A. GRUBER, A. M., War Department, Washington. D. C.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and HENRY HEATON, A. M., Atlantic, Iowa.

Let O be the center of the given circle, and P the given point in the given chord AB.



Through P draw EF perpendicular to chord AB. Draw the diameter COD parallel to EF.

Through C and P draw line CH terminating at H in the circumference of the given circle.

Draw OH intersecting EF at M.

Then will M be the center of a circle tangent to chord AB at P, and tangent to the given circle at H.

PROOF. The center of a circle tangent to AB at P must lie in the perpendicular EF.

The radii OH and MH are drawn to the point of tangency of the two circles. Therefore, the centers O and M and the point of tangency H must lie in the same straight line.

There remains to be proved $MP=MH$.

By construction, MP is parallel to OC, and $OC=OH$. Whence $\triangle COH$ and $\triangle PMH$ are similar.

$\therefore$ $MP=MH$, and M is the center of the required circle, MP and MH being radii thereof.

By a similar construction, we find N the center of a tangent circle on the other side of chord AB, the point of tangency being G.

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.; JOHN J. QUINN, Instructor in Mathematics, Rochester Athenaum and Mechanics Institute, Rochester, N. Y.; GAYLOR CAMERON, Student Heidelberg University, Tiffin, Ohio ; and P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.

Suppose the problem solved, and let O be the given circle, and E the center of the required circle, and C the given point. One locus of E is the perpendicular to AB at C. From C, on this perpendicular, take CD=radius of given circle ; then $EO=ED$. Hence another locus of E is HE, the perpendicular bisector of OE. The intersection of HE and CD determines E.

III. Solution by the PROPOSER.

Let AOB be the given circle, AB the chord, and P the given point, C the center of given circle.

At P erect perpendicular, and with CB as radius construct circle MPN tangent to AB at P. Draw MN intersecting perpendicular at C, which is center of the required circle.

Excellent solutions were received from *J. W. YOUNG, J. SCHEFFER, ELMER SCHUYLER, CHAS. C. CROSS, WALTER H. DRANE, ALOIS F. KOVARIK*, and *P. H. PHILBRICK*.

121. Proposed by AUGUSTUS J. REEF, Student in Illinois State Normal University, Carbondale, Ill.

Construct a triangle having given its three medians. [From Wentworth's *Plane and Solid Geometry.*]

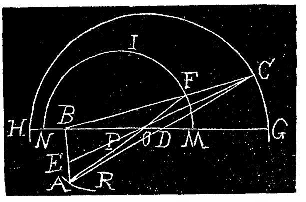
I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; ALOIS F. KOVARIK, Instructor in Mathematics, Decorah Institute, Decorah, Ia.; CHAS. C. CROSS, Whaleyville, Va.; and the PROPOSER.

Each median intersects the other medians at a common point two-thirds of the distance from the vertex to the middle of the opposite side.

Let AF, BD, and CE be the three medians of a triangle.

Trisect each of the medians.

Take any point O as a center, and with a radius equal to two-thirds of CE, the greatest median, describe the semi-circumference HCG.



Draw the diameter HOG.

With a radius equal to two-thirds of BD, the next largest median, and O as a center, intersect HOG at B.

Bisect, respectively, HB at N, BG at M, and MN at P.

Then, with NP, $=PM$, as a radius and P as a center, draw the indefinite arc $NIFM$. *This arc bisects any straight line drawn from point B to outer arc.*

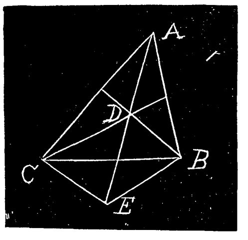
With a radius equal to one-third of median AF and O as a center, intersect arc $NIFM$ at F; and with a radius equal to two-thirds of AF and O as a center, describe the indefinite arc AR.

Through F and O draw line FA terminating in arc AR. Also draw lines BA and AC.

Then will ABC be the required triangle.

II. Solution by HENRY HEATON, M. Sc., Atlantic. Ia.; ELMER SCHUYLER, Reading, Pa.; J. D. CRAIG, A. B., New Germantown, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; GAYLOR CAMERON, Tiffin, O.; and W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Construct the triangle DEB such that DE, DB, and EB shall be, respectively, equal to two-thirds the given medians from the angles A, B, and C, of the required triangle.



Draw EC parallel to BD, and DC parallel to BE, meeting in C.

Prolong ED to A, making $DA=ED$. Join AB and AC.

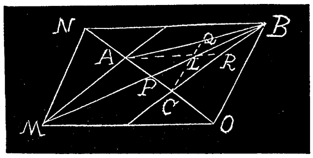
Then will ABC be the the required triangle.

The demonstration is obvious.

III. Solution by J. W. YOUNG, Columbus, Ohio, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Construct a parallelogram such that MN, MO, and MB shall be double the given medians.

Draw the other diagonal NO. Trisect NO in A, C. The triangle ABC is the one required, since PB is evidently one of the medians given, and the other medians QC and AR are, respectively, equal to $\frac{1}{2}OB$ and $\frac{1}{2}NB$. This is clear, from the considerations of the similar triangles AOB and AQC ($AQ=\frac{1}{2}AB$, $AC=\frac{1}{2}AO$, $\therefore$ $QC=\frac{1}{2}OB$), and NCB and ACR ($AC=\frac{1}{2}NC$, $RC=\frac{1}{2}BC$, $\therefore$ $AR=\frac{1}{2}NB$).



CALCULUS.

90. Proposed by ELMER SCHUYLER, Reading, Pa.

Prove that the evolute of the logarithmic spiral is an equal logarithmic spiral. [From Byerly's *Integral Calculus*.]

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; GEORGE LILLEY, Ph. D., L. L. D., Professor of Mathematics. State University, Eugene, Ore.; WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.; and ELMER SCHUYLER, Reading, Pa.

The intrinsic equation to the logarithmic spiral $s=k(c^t-1)$.

$ds/dt=kc^t\log c$, for the evolute $\sigma=\pm(ds/dt)]_o^t$.

$\therefore$ $\sigma=kc^t\log c-k\log c=k\log c(c^t-t)$.

$\therefore$ $\sigma=k'(c^t-1)$, an equal spiral.

II. Solution by J. SCHEFFER. A. M., Hagerstown, Md.; and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let P be a point of the given curve $r=a^\theta$, O the center of curvature, PQ a tangent at P, $PO=\rho$, $SQ=p=$the perpendicular from S upon PQ, $SP=r$, $SO=r'$, SM perpendicular to OP and$=p'$.

The pedal equation of the given curve $r=a^\theta$ is $r=p\sqrt{[1+(\log a)^2]}$; we also have $r'^2=\rho^2+r^2-2\rho p$, but $\rho=r\sqrt{[1+(\log a)^2]}$.

$\therefore$ $r'=r\log a$, and since $p'^2=r^2-p^2$, we have

$$p'^2=\frac{r^2(\log a)^2}{1+(\log a)^2}. \quad \therefore\ p'=\frac{r\log a}{\sqrt{[1+(\log a)^2]}}. \quad \therefore\ p'=\frac{r'}{\sqrt{[1+(\log a)^2]}},$$

or, $r'=p'\sqrt{[1+(\log a)^2]}$, which is the pedal equation of the evolute and exactly like the pedal equation of the logarithmic spiril.

III. Solution by CHAS. E. MYERS, Canton, Ohio; and P. H. PHILBRICK, M. S., C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.

Let $r=$the radius vector of the given curve, $p=$the perpendicular on the tangent, $r_1=$the radius vector of the evolute, $p_1=$the perpendicular on its tangent, and $R=$the radius of curvature.

We have for the curve, $r=cp$......(1).

$\therefore\ dr/dp=c$, and $R=cr$......(2).

$$r^2=p^2+p_1^2\ \ .....(3).\quad r_1^2=R^2+r^2-2Rp......(4).$$

From (1), (2), and (4), we have,

$$r_1^2=c^2p^2(1+c^2)-2c^2p^2=c^2p^2(c^2-1)......(5).$$

From (3), and (1),

$$c^2p^2=p^2+p_1^2,\ \text{or},\ p^2(c^2-1)=p_1^2......(6).$$

From (5), and (6),

$$r_1^2=c^2p_1^2,\ \text{or},\ r_1=cp_1,$$

the equation of a similar and equal logarithmic spiral.

IV. Solution by J. W. YOUNG, Graduate Student, Ohio State University, Columbus, Ohio.

The equation to the logarithmic spiral is

$$r=ae^{m\theta}.$$

The equation to the normal at the point $(r_1,\ \theta_1)$ on the spiral is

$$r\cos[\theta-(\theta_1+\phi)]=r_1\cos\phi......(1),$$

where ϕ is the angle between r_1 and the tangent, and ϕ is constant, a known property of the spiral.

Expanding the left-hand member of (1), we have

$$r[\cos\theta(\cos\theta_1\cos\phi-\sin\theta_1\sin\phi)+\sin\theta(\sin\theta_1\cos\phi+\cos\theta_1\sin\phi)=r_1\cos\phi.$$

Dividing through by $\cos\phi$, and remembering that $\tan\phi$=constant, which can easily be shown to be $1/m$, we have

$$r\{\cos\theta[\cos\theta_1-(1/m)\sin\theta_1]+\sin\theta[\sin\theta_1+(1/m)\cos\theta_1]=r_1=ae^{m\theta_1}.$$

The required evolute is the envelope of this line where θ_1 is the parameter. Hence, arranging in terms of functions of θ_1, we have

$$r\{\cos\theta_1[\cos\theta+(1/m)\sin\theta]+\sin\theta_1[\sin\theta-(1/m)\cos\theta]\}=ae^{m\theta}......(2).$$

Differentiating with respect to θ_1,

$$r\{-\sin\theta_1[\cos\theta+(1/m)\sin\theta]+\cos\theta_1[\sin\theta-(1/m)\cos\theta]\}=ame^{m\theta_1}......(3).$$

We must eliminate θ_1 between (2) and (3).

Dividing (2) by (3), we have

$$\frac{\cos\theta_1(m\cos\theta+\sin\theta)+\sin\theta_1(m\sin\theta-\cos\theta)}{-\sin\theta_1(m\cos\theta+\sin\theta)+\cos\theta_1(m\sin\theta-\cos\theta)}=\frac{1}{m}.$$

Reducing, $\cos\theta_1.\cos\theta=-\sin\theta_1\sin\theta$, or

$$\frac{\cos\theta_1}{\sin\theta}=-\frac{\sin\theta_1}{\cos\theta}=\frac{\sqrt{(\cos^2\theta_1+\sin^2\theta_1)}}{\sqrt{(\sin^2\theta+\cos^2\theta)}}=1.$$

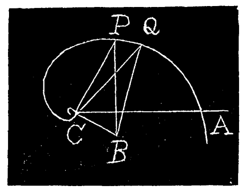
Hence, $\cos\theta_1=\sin\theta$, $\sin\theta_1=-\cos\theta$.

$\therefore$ $\theta_1=\theta-\frac{1}{2}\pi$.

$\therefore$ The required equation is $r=ae^{m(\theta-\frac{1}{2}\pi)}=a'e^{m\theta}$.

V. **Solution by P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.**

Let P and Q be two points on the curve, C its pole, PB, QB, the normals at P and Q; join CB. Then the angles CPB and CQB are equal, and consequently the four points C, P, Q, B, lie on a circle. Hence $\angle QCB=\angle QPB$; but in the limit when P and Q are coincident, the angle QPB becomes a right angle and B becomes the center of curvature belonging to point P; hence PCB also becomes a right angle, and the point B is determined.



Again, $\angle CBP=\angle CQP$; but, in the limit, the angle CQP is constant; therefore angle CBP is also constant, and since the line BP is a tangent to the evolute at B, it follows that the tangent makes a constant angle with the radius vector CB. From this property it follows that the evolute in question is another logarithmic spiral.

Again, as the constant angle is the same for the curve and for its evolute, it follows that the latter curve is the same spiral.

Professors M. C. Stevens and Elmer Schuyler refer to the solution in *Williamson's Differential Calculus*, page 300.

MECHANICS.

Note on Problem 82.

Professor Zerr, in his solution of this problem, assumed limiting friction at all points. Of course there is limiting friction between box and floor. But limiting friction does not in general exist at both the other two points; for suppose it did, then the sphere in its descent will not revolve about its center.

Hence, taking moments about the center (using his notation) we have

$$\tfrac{1}{3}W'=\frac{2W'}{3\cos\theta+9\sin\theta}.$$

Whence $\theta=60^\circ$, no matter what the relation between W and W'. Of

course W and W' might be so related that this would be true, but in general they are not. To see at which point there is limiting friction, we might proceed as follows: Solve on the supposition that limiting friction exists between sphere and box, and not between sphere and wall; then solve with the opposite assumption; one of these results will be found smaller than the other, I think it is the latter; we infer then that limiting friction must exist at wall and not at box, and hence that the latter result is the correct one. W. H. DRANE.

88. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics, Decorah Institute, Decorah, Ia.

Show that the equation to the trajectory is

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha},$$

and that v and a can be varied at pleasure, the projectile can in general be made to traverse any two given points in the same vertical plane with the point of projection. [Ex. 83, page 244, Deschanel's *Natural Philosophy*, Part I.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; B. F. SINE, Principal of Normal School, Rock Enon Bridge, W. Va.; ELMER SCHUYLER, Reading, Pa.; and the PROPOSER.

Let v=velocity of projection, α=angle of elevation, t=time, (x, y) the coördinates of the point in its path at the time t.

$\therefore$ $x=vt\cos\alpha$=horizontal motion. $y=vt\sin\alpha-\frac{1}{2}gt^2$=vertical motion.

Eliminating t, we get at once,

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha}\ldots\ldots(1).$$

(1) is true no matter what be the values of v and α.

Let (m, n), (b, c) be the coördinates of two points. Then from (1) we get

$$n=m\tan\alpha-\frac{gm^2}{2v^2\cos^2\alpha},\quad c=b\tan\alpha-\frac{gb^2}{2v^2\cos^2\alpha}.$$

$$\therefore\ \alpha=\tan^{-1}\left(\frac{nb^2-m^2c}{mb^2-m^2b}\right),\quad v^2=\frac{g[(mb^2-m^2b)^2+(nb^2-m^2c)^2]}{2(mb^2-m^2b)(bn-mc)}.$$

These values of α and v will cause the trajectory to pass through the two given points.

89. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Assuming that the Northern Pacific R. R. tracks between Fargo and Bismark (North Dakota) to lie on the 47th parallel of latitude; also that the Limited Express weighs 300 tons, and that a speed of 60 miles per hour is maintained between the two places· find the difference between the vertical pressures on the rails of the Express east and the express west.

I. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Centrifugal force $=mv^2/\rho_1$.

Taking $g=32$ we have $m=18750$ for mass of train.

For 47th parallel of latitude $\rho=14403840$ feet, and $v=1048.89$ feet per second is velocity of a point at that parallel.

The velocity of the train on the surface of the earth is 88 feet per second.

$\therefore$ $1048.89-88=960.89$ feet is velocity in space of train going west.

$1048.89+88=1136.89$ feet is velocity in space of train going east.

For train going west we have for centrifugal force,

$$\frac{mv^2}{\rho}=\frac{18750(960.89)^2}{14403840}=1201.1059 \text{ pounds.}$$

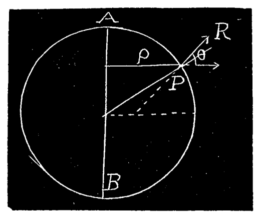
For train going east,

$$\frac{mv^2}{\rho}=\frac{18750(1136.89)^2}{14403840}=1682.1923 \text{ pounds.}$$

$1682.1923-1201.1059=481.0864$ pounds, amount by which the decrease in weight of train going east exceeds the decrease in weight of train going west.

If by vertical pressure is meant that towards center of circle of latitude, the above is the excess in pressure of train going west over that of train going east.

If the pressure is understood to act towards the center of the earth the difference is $(481.0864)\cos 47^\circ=328.6312$ pounds.



If however, the difference in the weight of the two trains is required, we must proceed as follows:

We have assumed the earth to be a perfect sphere. On account of the motion of the earth the apparent line of weight is slightly deflected from the vertical. Let PR be the line of action of apparent weight, $R=mg$ at a point P on the earth's surface. Let G be the value gravity would have if the earth were still, and then the force along PO will be mG. Let φ be the complement of the latitude of P. Resolving along CP and perpendicular to it we get

$$\frac{mv^2}{\rho}=mG\sin\varphi-mg\cos\theta\ldots\ldots(1).$$

$$mg\sin\theta=mG\cos\varphi\ldots\ldots(2).$$

Eliminating θ from (1) and (2) and solving for G gives

$$G=\frac{v^2\sin\varphi}{\rho}\pm\sqrt{g^2-\frac{v^4}{\rho^2}\cos^2\varphi}.$$

Substituting values for the case in hand we find $G=32.05209$.

From (2),

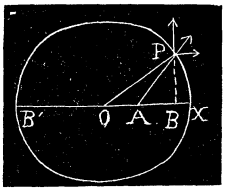
$$\sin\theta = \frac{G\cos\varphi}{g} = \frac{32.05209\times.731}{32} = .73219.$$

$\therefore$ $\theta=47^\circ\ 4'\ 15''$.

$\therefore$ Difference in apparent weight$=481.0864\cos(47^\circ\ 4'\ 15'')=327.6583$ lbs.

II Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

This problem depends on the velocities in space of the Express east and the Express west.



In the figure, let P be the point of the train on the 47th parallel, O the center of the earth, AP the normal at P, φ the angle PO makes with the equatorial diameter $B'x$, θ the angle the normal makes with the same line, (θ is the latitude of P).

Let $OB=\rho$, $OP=r$, 300 tons$=W$, $f=$centrifugal force in the direction AP (vertical direction), $g=$gravity on 47th parallel, $G=$gravity at equator. $a=6377377$ meters$=20923536$ feet$=$equatorial radius, $e=$ellipticity of the earth.

$\rho=r\cos\varphi=a\cos\theta/\sqrt{(1-e^2\sin^2\theta)}$.

Now $\theta=47^\circ$. $e^2=.006920928$. $\therefore$ $\rho=4357445.45$ meters.

One day$=86400$ seconds.

$\therefore$ $2\pi\rho/86400=316.8831$ meters$=1039.37$ feet per second, the velocity of P due to the earth's rotation.

60 miles an hour$=88$ feet per second.

$1039.37-88=951.37$, the train's velocity in space going west.

$1039.37+88=1127.37$, the train's velocity in space going east.

$f=Wv^2/g\rho=Wv^2/gr\cos\varphi$.

$\therefore$ $F=f\cos\theta=Wv^2\sqrt{(1-e^2\sin^2\theta)}/ag$.

Now $g=G(1+\frac{1}{4}e^2\sin^2\theta)$.* $G=32.2015235$ feet.

$\therefore$, $g=32.23130991$ feet per second. $\therefore$ $F=.000000444v^2$ tons.

$\therefore$ $F=.4018399$ tons going west. $F=.5643076$ tons going east.

$\therefore$ Difference$=.1624657$ tons$=324.9354$ pounds.

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*This expression for the true value of gravity in latitude θ is new to me. If any reader of the MONTHLY can tell me where to find it used previously, and by whom, I would be greatly pleased to know. I believe it to be new and unused before.

AVERAGE AND PROBABILITY.

76. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

In a given ellipse, the extremities of a focal chord are joined with the center. Find the average area of the angle thus formed.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metalurgy, Rolla, Mo.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; J. SCHEFFER, A. M., Hagerstown, Md.; and L. C. WALKER, Instructor in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

Let θ=the angle the focal chord makes with the major axis.

The length, l, of this focal chord

$$=\frac{a(1-e)^2}{1+e\cos\theta}+\frac{a(1-e^2)}{1+e\cos\theta}=\frac{2a(1-e^2)}{1-e^2\cos^2\theta}.$$

The perpendicular distance from the center to this chord$=ae\sin\theta$.

$\therefore$ Area of triangle=

$$A=\frac{a^2e(1-e^2)\sin\theta}{1-e^2\cos^2\theta}.$$

I. When the chords are drawn at equal angular intervals,

$$\triangle=\frac{\int_0^{\frac{1}{2}\pi}Ad\theta}{\int_0^{\frac{1}{2}\pi}d\theta}=\frac{2a^2e(1-e^2)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\theta d\theta}{1-e^2\sin^2\theta}=\frac{a^2(1-e^2)}{\pi}\log\left(\frac{1+e}{1-e}\right).$$

DEAN, DRANE, SCHEFFER, WALKER.

II. When the abscissas of the extremity of the chord are drawn at equal intervals,

$$\triangle=\frac{\int Adx}{\int dx}=\frac{\int_0^{\frac{1}{2}\pi}\frac{A\sin\theta d\theta}{(1-e\cos\theta)^2}}{\int_0^{\frac{1}{2}\pi}\frac{\sin\theta d\theta}{(1-e\cos\theta)^2}}=a^2e(1-e)^2(1+e)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-e\cos\theta)^3(1+e\cos\theta)}$$

$$=-\tfrac{1}{2}a^2(1-e)^2(1+e)+\tfrac{1}{2}a^2(1-e)^2(1+e)\int_0^{\frac{1}{2}\pi}\frac{(e+\cos\theta)d\theta}{(1-e^2\cos^2\theta)^2}$$

$$=\frac{a^2(1-e)}{8b}\left(\pi a^3e^3-4b(1-e^2)+20a^2\sqrt{1-e^2}+\frac{2a^2}{e}\sin^{-1}e\right).$$

III. When the chord varies with the arc,

$$\triangle=\frac{\int Ads}{\int ds}.$$

ZERR.

77. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.; and ELMER SCHUYLER, Reading, Pa.

A and B are two inaccurate mathematicians whose chance of solving a given question correctly is 1/8 and 1/12 respectively; if they obtain the same result, and if it is 1000 to 1 against their making the same mistake, find the chance that the result is correct. [From *Hall and Knight's Algebra.*]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; COOPER D. SCHMITT, A. M., Ph. D., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and L. C. WALKER, Instructor in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

The chance that A and B both get the correct result is $\frac{1}{8}\times\frac{1}{12}=\frac{1}{96}$.

The chance that both get the wrong result is $\frac{7}{8}\times\frac{11}{12}=\frac{77}{96}$.

The chance that they both get the *same* wrong result is $\frac{1}{1001}\times\frac{77}{96}=\frac{1}{13.96}$ $=\frac{1}{1248}$.

$\therefore$ The chance that the result is correct : the chance that the result is not correct :: 13 : 1.

$\therefore$ The required chance is $\frac{13}{14}$.

Also the required chance is $(\frac{1}{96})/(\frac{1}{96}+\frac{1}{1248})=\frac{13}{14}$.

78. Proposed by CHAS. E. MYERS, Canton, O.

Two witnesses, A and B, both make the statement that an event happened in a particular way (two ways being possible). Find the probability of the truth of the statement.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science. Chester High School, Chester, Pa.

Let p, q be the chances that A, B speak the truth, respectively.

Then the chance of the truth of the statement is

$$\frac{pq}{pq+(1-p)(1-q)}=c.$$

Now $p=q=\frac{1}{2}$. $\therefore c=\frac{1}{4}/[\frac{1}{4}+(1-\frac{1}{2})(1-\frac{1}{2})]=\frac{1}{4}/\frac{1}{2}=\frac{1}{2}$.

MISCELLANEOUS.

72. Proposed by E. D. ROE. Jr., A. M., Ph. D., Professor of Mathematics. Oberlin College, Oberlin, Ohio.

If a, b, and c are integers, and

$$\left\{\begin{matrix} b,\ c-b,\ c-1 \\ c-a-1 \\ c-a-1 \end{matrix}\right\}>0,$$

prove that the sum of the series,

$$1+\frac{a.b}{1.c}+\frac{a(a+b).b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}+\ldots\ldots$$

is equal to

$$\frac{(c-1)!\,(c-a-b-1)!}{(c-a-1)!\,(c-b-1)!}.$$

I. **Solution by the PROPOSER.**

Let $fx=\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du.$

If $0<x<1$, $0\leqq u\leqq 1$, $(1-xu)^{-a}=1+\frac{a}{1}xu+\frac{a(a+1)}{1.2}x^2u^2$

$$+\frac{a(a+1)(a+2)}{1.2.3}x^3u^3+\ldots\ldots$$

and therefore

$$fx=\int_0^1 u^{b-1}(1-u)^{c-b-1}du+\frac{a}{1}x\int_0^1 u^b(1-u)^{c-b-1}du$$

$$+\frac{a(a+1)}{1.2}x^2\int_0^1 u^{b+1}(1-u)^{c-b-1}du+\ldots\ldots$$

$$=B(b,\ c-b)+\frac{a}{1}xB(b+1,\ c-b)+\frac{a(a+1)}{1.2}x^2B(b+2,\ c-b)+\ldots\ldots(1),$$

where $B(m,n)=\int_0^1 u^{m-1}(1-u)^{n-1}du$, is known as the Beta Function or First Eulerian Integral. (Cf. Byerly's *Integral Calculus*, page 109.)

Now we have l. c. page 110,

$$B(m,\ n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \text{with}\ \Gamma(n+1)=n\Gamma(n),\ \text{where}\ \Gamma(n)=\int_0^\infty u^{n-1}e^{-u}du,$$

and is known as the Gamma Function or Second Eulerian Integral.

We have therefore,

$$B(m+1,\ n)=\frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}=\frac{m}{m+n}B(m,n).$$

If in this formula $m=b$, $n=c-b$, we get $B(b+1,\ c-b)=(b/c)B(b,\ c-b)$.

Similarly,

$$B(b+2,\ c-b)=\frac{b+1}{c+1}B(b+1,\ c-b)=\frac{b(b+1)}{c(c+1)}B(b,\ c-b).$$

$$B(b+3,\ c-b)=\frac{b+2}{c+2}B(b+2,\ c-b)=\frac{b(b+1)(b+2)}{c(c+1)(c+1)}B(b,\ c-b),$$

and by mathematical induction,

$$B(b+n,\ c-b)=\frac{b(b+1)\ldots\ldots(b+n)}{c(c+1)\ldots\ldots(c+n)}B(b,\ c-b).$$

By substituting these values in (1), we get

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du=B(b,\ c-b)\Big(1+\frac{a.b}{1.c}x+\frac{a(a+1).b(b+1)}{1.2\ \ c(c+1)}x^2$$

$$+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ \ c(c+1)(c+2)}x^2+\ldots\ldots\Big)\ldots\ldots(2).$$

Taking the limit of both members of (2) as $x\doteq1$, we have

$$\Big[\int_0^1 u^{b-1}(1-u)^{c-a-b-1}du=\Big]B(b,\ c-a-b)=B(b,\ c-b)\Big[1+\frac{a.b}{1.c}$$

$$+\frac{a(a+1).\ b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ \ c(c+1)(c+2)}+\ \ \ldots\ldots\Big]\ldots\ldots(3),$$

or $\Big[1+\frac{a.b}{1.c}+\frac{a(a+1).b(b+1)}{1.2\ \ c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ \ c(c+1)(c+2)}+\ldots\ldots\Big]$

$$=\frac{B(b,\ c-a-b)}{B(b,\ c-b)}=\frac{\Gamma(b)\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(b)\Gamma(c-b)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$=\frac{(c-1)!\ (c-a-b-1)!}{(c-a-1)!\ (c-b-1)!}\ \ldots\ldots(4),$$

if a, b, and c are integers, and the inequalities stated in the problem are satisfied.

II Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $y_{cx}=1+\frac{ab}{1.c}x+\frac{a[a+1]b[b+1]}{1.2.c[c+1]}x^2+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.3.c[c+1][c+1]}x^3+\ldots\ldots$

$$\therefore\frac{dy_{cx}}{dx}=\frac{ab}{1.c}+\frac{a[a+1]b[b+1]}{1.c[c+1]}x+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.c[c+1][c+2]}x^2+\ldots\ldots$$

$$=\frac{ab}{c}\Big[1+\frac{[a+1][b+]}{1\ [c+1]}x+\frac{[a+1][a+2]\ b[b+1][b+2]}{1.2.c[c+1][c+2]}x^2+\ \ldots\ldots\Big]$$

$$\frac{d^2y_{cx}}{dx^2}=\frac{a[a+1]b[b+1]}{c.[c+1]}\Big[1+\frac{[a+2][b+2]}{2.[c+2]}x+\ \ldots\ldots\Big]$$

$$\therefore\ x(1-x)\frac{d^2y_{cy}}{dx^2}+\{c-[a+b+1]x\}\frac{dy_{cx}}{dx}=aby_{cx}\ldots\ldots(1).$$

$$y_{(c-1)x}=1+\frac{ab}{1.[c-1]}x+\frac{a[a+1]b[b+1]}{1.2.[c-1]c}x^2+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.3.[c-1]c[c+1]}x^3+\,..$$

$$y_{cx}-y_{(c-1)x}=-\frac{abx}{c[c-1]}\left[1+\frac{[a+1][b+1]}{1.[c+1]}x+\frac{[a+1][a+2][b+1][b+2]}{1.2.[c+1][c+2]}x^2+.\right]$$

$$\therefore\ y_{(c-1)x}-y_{cx}=\frac{abx}{c[c-1]}\cdot\frac{c}{ab}\frac{dy_{cx}}{dx}=\frac{x}{c-1}\frac{dy_{cx}}{dx}\ldots\ldots(2).$$

If we make x=unity in (1) we get

$$\frac{dy_c}{dx}=\frac{aby_c}{c-a-b-1}.\qquad\therefore\ y_{c-1}-y_c=\frac{aby_c}{[c-1][c-a-b-1]}.$$

$$\therefore\ y_{c-1}=\frac{[c-1][c-a-b-1]+ab}{[c-1][c-a-b-1]}y_c=\frac{[c-a-1][c-b-1]}{[c-1][c-a-b-1)}y_c.$$

By symmetry,

$$y_c=\frac{[c-a][c-b]}{c[c-a-b]}y_{c+1},\quad y_{c+1}=\frac{[c-a+1][c-b+1]}{[c+1][c-a-b+1]}y_{c+2}\ldots\ldots(3,\ 4).$$

(4) in (3) gives

$$y_c=\frac{[c-a][c-a+1][c-b][c-b+1]}{c[c+1][c-a-b][c-a-b+1]}y_{c+2}.$$

$$\therefore\ y_c=\frac{[c-a][c-a+1]\ldots.[c-a-1+n][c-b][c-b+1]\ldots.[c-b-1+n]}{c[c+1]\ldots.[c-1+n][c-a-b][c-a-b+1]\ldots.[c-a-b-1+n]}y_{c+n}.$$

Let $[c-a-1]=s$, $[c-b-1]=t$, $[c-1]=u$, $[c-a-b-1]=v$. Then

$$_c=\frac{\dfrac{[s+1][s+2]\ \ldots\ldots.[s+n]}{1.2.3\ \ldots.\ n}\cdot\dfrac{[t+1][t+2]\ldots\ldots..[t+n]}{1.2.3\ldots..\ n}}{\dfrac{[u+1][u+2]\ldots\ldots\ [u+n]}{1.2.3\ldots\ldots n}\cdot\dfrac{[v+1][v+2]\ldots\ldots\ [v+n]}{1.2.3\ \ldots..n}}y_{c+n}$$

$$=\frac{\dfrac{[1+(1/n)][1+(2/n)]\,..\,[1+(s/n)]}{1.2.3\ldots\ldots s}\cdot\dfrac{[1+(1/n)][1+(2/n)]\ldots[1+(t/n)]}{1.2.3\ldots\ldots t}}{\dfrac{[1+(1/n)][1+(2/n)]\ldots[1+(u/n)]}{1.2.3\ldots\ldots u}\cdot\dfrac{[1+(1/n)][1+(2/n)]\,..\,[1+(v/n)]}{1.2.3\ldots\ldots v}}y_{c+n}.$$

Let $n=\infty$, then $y_{c+n}=1+\dfrac{ab}{c+\infty}+\dfrac{a[a+1]b[b+1)}{1.2[c+\infty][c+\infty+1]}+\ldots\ldots$

$\therefore\ y_{c+n}=1.$

$\therefore\ y_c=\dfrac{u!\,v!}{s!\,t!}=\dfrac{[c-1]!\ [c-a-b-1]!}{[c-a-1]!\ [c-b-1]!}.$

But $y_c=1+\dfrac{ab}{1.c}+\dfrac{a[a+1]b[b+1]}{1.2.c[c+1]}+\ldots\ldots$

Therefore. etc. [See *Forsyth's Differential Equations*, chapter VI, page 185, for treatment of this series.]

PROBLEMS FOR SOLUTION.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Suppose 10% traction stock is 20% better in the market than 5% mining stock; if my income be $500 from each, how much money have I paid for each, the whole investment bringing 6⅔%?

123. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

If $m=2$ cents be the interest on $M=100$ cents for $p=40$ days, find the yearly rate per cent.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than January 10.

ALGEBRA.

111. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equation $x(y+z)=a(x+y+z)$, $y(x+z)=b(x+y+z)$, $z(x+y)=c(x+y+z)$.

112. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In *Hall and Knight's Higher Algebra*, I find the following:

If $a+b+c=0$, then

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2};\qquad \frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2};$$

also if $a+b+c+d=0$, then

$$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

QUERY. Is there a general law governing such expressions? Investigate.

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

GEOMETRY.

131. Proposed by J. W. YOUNG. Graduate Student, Ohio State University, Columbus, Ohio.

Prove that $\lambda+\mu\omega+\nu\omega^2$, where λ, μ, ν are integers whose sum is ±1, represents the points of a quilt formed by regular hexagons. ω=primitive cube root of unity. [From *Harkness and Morley's Introduction to Theory of Functions.*]

132. Proposed by ELMER SCHUYLER, Reading, Pa.

To draw a circle to cut two given circles orthogonally.

133. Proposed by P. C. CULLEN, Principal of Public Schools, Indianola, Neb.

If the two bisectors, trisectors, quadrisectors, etc., of the base angles of a triangle are mutually equal, show that the triangle is isosceles.

134. Proposed by J. C. CREGG, A. M., Superintendent of Schools, Brazil, Ind.

If $ABCD$ is a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals AB, CD passes through the center of the circle.

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

CALCULUS.

102. Proposed by G. B. M. ZERR, A.M., Ph.D.. Professor of Mathematics and Science, Chester High School, Chester, Pa.

A right cone has its vertex at the focus of a paraboloid of revolution, the axis of the cone perpendicular to the axis of the paraboloid. Find the volume common to both.

103. Proposed by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A park, in the shape of an ellipse whose diameters are 100 and 50 rods, respectively, is surrounded by a wall: one end of a rope, whose length is the circumference of the ellipse, is fastened (outside of the wall) at one end of the longer diameter and the other end at the otoer end of the same diameter. Over how much surface will a horse graze, which is fastened to a ring moving freely on the rope?

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

AVERAGE AND PROBABILITY.

85. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Two points are taken at random in a circle and a chord drawn through them; a point is then taken at random in each segment. Find the average area of the quadrilateral formed by joining the four points.

86. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

Two points are taken at random in a circular annulus formed by two concentric circles. Find the chance that the straight line joining the points will not cut the inner variable circle.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the mean distance of a random point in a sphere from a point, (1) within, (2) without the sphere.

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

EDITORIALS.

Prof. James Pierpont, of Yale University, was elected a member of the French Mathematical Society.

Dr. Artemas Martin, of the United States Coast and Geodetic Survey, at Washington, D. C., has recently been elected a member of the Deutsche Mathematiker-Vereinigung.

At the recent meeting of the Board of Trustees of the Indiana University, Dr. John B. Faught, Instructor in Mathematics, was given the title of Assistant Professor of Mathematics.

Prof. J. H. Tanner, of Cornell University, is writing an elementary algebra for the "Cornell Series," and Prof. James McMahon is writing an elementary geometry for the same series.

We are pleased to state that Dr. Halsted is receiving many congratulatory letters from readers of the MONTHLY who are reading with interest and profit, his contributions to the MONTHLY in Non-Euclidean Geometry.

Through the kindness of Dr. Alexander Macfarlane, we have received for publication a paper on "The Theory of Mathematical Inference," by Prof. G. J. Stokes, of Queen's College, Cork, Ireland. This paper will appear in the January number, Vol. VII, of the MONTHLY.

A number of our readers have asked us to send them the July issue. In answer we will say, take the cover off the June number and you will then have a June-July number. Through error on the part of the printers the statement of a double number was omitted on the cover.

BOOKS AND PERIODICALS.

The Use of the Slede Rule. By F. A. Halsey, Associate Editor "American Machinist," Consulting Engineer Rand Drill Co. 16mo. 84 pages. Price, 50 cents. New York: D. Van Nostrand Co.

In this little book the author has set forth very clearly the use of the Slede Rule and has thus rendered valuable service to the practical computer. B. F. F.

Grammar School Algebra. By William J. Milne, Ph. D., LL. D., President of New York State Normal College, Albany, N. Y. 154 pages. Price, 50 cents. 1899. New York, Cincinnati, and Chicago: American Book Company.

This book is characterized by the same methods of presentation exemplified in the other books of Dr. Milne's series. It will meet with hearty approval. J. M. C.

Standard School Algebra. By George E. Atwood. 432 pages. Price, $1.20. 1898. New York: The Morse Company.

This book is designed for use in high schools, and academies, and advanced classes in grammar schools. The definitions, demonstration of principles, derivation of rules, model solutions, and illustrations, occupy the last half of the book, and the exercises and problems in the first half. This arrangement may commend itself to many, but we fail to see any real advantage to be gained by reducing the first part of the book to a bare collection of exercises and problems. In other respects the book is highly satisfactory. The second part is marked by a clearness and conciseness in definitions, careful demonstration of principles, and an abundance of illustrations and model solutions. The book satisfactorily meets the requirements of what is best in the science and method of teaching elementary algebra at the present time. J. M. C.

La Mathématique Philosophie—Enseignement. Par C. A. Laisant, Répétiteur a L'École Polytechnique Docteur ès Sciences. 8vo. Cloth, 292 pages. Price, $1.25. Paris: Georges Carré et C. Naud.

In this work the author has considered the philosophy and teaching of mathematics in a way so as to be of service not only to the student of mathematics, but to teachers as well. In No. 4, Vol. V, of the MONTHLY, Dr. Alexander Macfarlane contributed some remarks *in extenso*, from this book. Professor Laisant, in the remarks referred to, says that during the last twenty-five years, few countries have made greater progress in mathematics than the United States. This remark some of our readers considered a jest on the part of the author. But such is not the case. The book is from first to last a most carefully and sincerely written work, intended to be of the highest service to students and teachers of mathematics. The book is divided into three parts: The first part discusses the philosophy of Pure Mathematics; the second part discusses the philosophy of Applied Mathematics; and the third part treats of the Instruction in Mathematics. The first part contains eight chapters, the first of which has to do with Mathematics and its subdivisions; the second with Arithmetic and the Theory of Numbers, or Arithmologie; the third, Algebra; the fourth, the Infinitesimal Calculus; the fifth, Theory of Functions; the sixth, Geometry; the seventh, Analytical Geometry; the eighth, Rational Mechanics. In the second part, chapter 1 is devoted to general considerations; chapter 2, to Applications of the Calculus; chapter 3, to Applications of Geometry; chapter 4, to Applications of Mechanics. In the third part, chapter 1 is devoted to a General View on the Teaching of Mathematics; chapter 2, Teaching of Arithmetic; chapter 3, Teaching of Algebra and the Advanced Calculus; chapter 4, Teaching of Geometry; chapter 5, teaching of Analytical Geometry; chapter 6, Teaching of Mechanics; chapter 7, The Hierarchy of Teaching.

B. F. F.

A Short Table of Integrals. Revised Edition. By Benj. Osgood Peirce, Ph. D., Hollis Professor of Mathematics and Natural Philosophy in Harvard University. 8vo. Cloth. 134 pages. Price, $1.00. Boston : Ginn & Co.

This little book in its revised and enlarged form, contains nearly all the integrals commonly needed by students of the elements of the integral calculus in American colleges. A number of pages of auxiliary formulas, involving trigonometric, hyperbolic, and elliptic functions, and useful in transforming and interpreting integrals, have been added, with a few numerical tables in which are given, though in fine type, the four-place logarithms of natural numbers and of the trigonometric functions, the values of the hyperbolic functions, etc. This book will qe found very serviceable to use with any text-book.

J. M. C.

The United States Sinking-Fund. By Theodore L. DeLand, Office of the Secretary of the Treasury, Washington. 1899.

We are indebted to Mr. DeLand for a copy of his solution, equation $u_{x+1}-u_x = r_1(a-u_x)+ru_x$, from advance sheets of Vol. II., No. 12, of the *Mathematical Magazine.*

J. M. C.

Observational Geometry. By William T. Campbell, A. M., Instructor in Mathematics in the Boston Latin School. With an Introduction by Andrew W. Phillips, Ph. D., Professor of Mathematics in Yale University. Over 300 Illustrations and Diagrams. 8vo., 240 pages. New York and London : Harper & Brothers. 1899.

The reasoning required in this book depends on direct observation and the measurement of geometric figures constructed by the pupils themselves. Part I. treats of elementary forms, beginning with the cube, and introduces at once the ideas of precision and accuracy. Part II. takes up geometric forms in a more minute manner, developing the ideas of arrangement, order, and symmetry. The matter has been skilfully and clearly presented, the illustations are exceedingly helpful, and the book in all its details seems to have been carefully and honestly written. As an introduction to the study of geometry for pupils of the upper grammar grades, it is perhaps the best book that has yet appeared.

J. M. C.

The Story of the Philippines and Our New Possessions, Including the Ladrones, Hawaii, Cuba, and Porto Rico. By Murat Halsted. Sold by the Dominion Co., Chicago, Ill.

This book is beautifully illustrated with half-tone engravings from photographs, etchings from special drawings, etc. The book is nicely bound, and there is a great demand for it, it being written by one of our ablest journalists. B. F. F.

The following periodicals have been received : *The American Journal of Mathematics*, October, 1899 ; *The Educational Times*, October 1, 1899 ; *Journal de Mathématiques Élémentaires*, 15 October, 1899 ; *The Monist*, October, 1899 ; *Bulletin of the American Mathematical Society*, July, 1899 ; *L'Intermédiaire des Mathématiciens*, Juillet, 1899 ; *The Kansas University Quarterly*, April, 1899 ; *The Mathematical Gazette*, June, 1899 ; *Mathematisch-naturwissenschaftliche Mitteilungen*, Oktober, *Herausgegeben* von Dr. O. Böklen and Dr. E. Wölffing, Stuttgart, Germany.

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SOME LOCI AND THEIR PROJECTIONS.

By ARTHUR R. CRATHORNE, B. S., University of Maine.

If a conic be described through two fixed points, A and B, and touching two given conics which also pass through those points, the locus of the of pole AB with respect to the varying conic is a conic touching the four lines CA, CB, $C'A$, and $C'B$, where C and C' are the poles of AB with respect to the two given conics.

At first reading this proposition or problem seems very complicated and difficult of solution ; and, indeed, it would be if the ordinary methods of analytical geometry were used. But treated from the standpoint of projective or the so-called modern geometry, the solution is very easy and leads to some interesting results. This proposition and the one deduced from it later in this article are but examples of a class which lends itself very easily to projective methods.

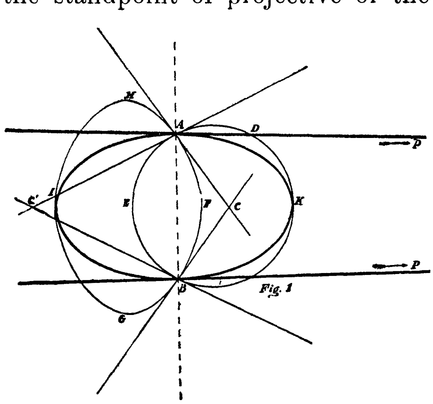

Fig. 1

In figure 1, let A and B be the two given points. Let the two fixed conics, $ADKB$ (called S) and $AGIH$ (called S') intersect at the given points. There will then be a doubly infinite number of conics which may pass through the two given points and be tangent to the conics S and S'. In figure 1, the heavy lined conic $AKBI$ is one of these conics. (There will be a set which will be tangent at about the points E and F). Draw the tangents AP

and BP. The point of intersection, P, will be the pole of the variable conic with respect to the line AB. As the conic $AKBI$ takes each of its infinite number of positions, the point P will trace a curve which will touch the tangent lines CA, $C'A$, CB, and $C'B$.

Let us project A and B into the focoids or circular points at infinity (*Scott's Modern Analytical Geometry*, Art 201). The line AB will be projected into the line at infinity (*Salmon's Conic Sections*, Art. 254). Now since a circle is the only conic which cuts the line at infinity at the focoids, the conics S, S' and $AKBI$ must be projected into circles when the points A and B are projected intc the focoids (*Scott's Modern Analytical Geometry*, Arts. 117-118). The poles of the line AB will now be at the centers of these circles (*Smith's Conic Sections*, Art. 314). Hence, after projection, the above proposition would read: "The locus of the centers of circles tangent to two given circles is a conic tangent to CA, CB, $C'A$, and $C'B$."

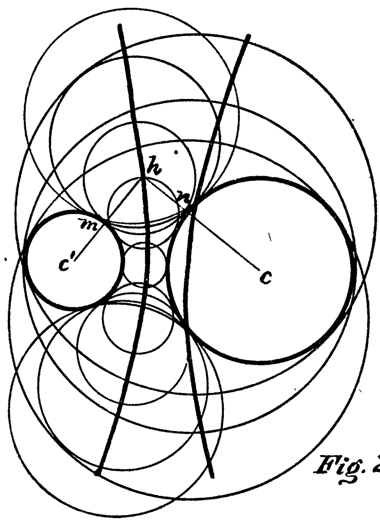

Fig. 2

In figure 2, let the heavy circles be the two given ones (or, the ones into which S and S' are projected). There will be four sets of tangent circles. Figure 2 shows those which lie outside of both the given circles and those which include both circles in their areas. The locus of the centers of these two sets will be an hyperbola, one set making one branch and the second set making the other. The centers of the given circles are the foci of the locus. That this curve is an hyperbola may be easily proved. Let h (Fig. 2) be any point on the curve; let r and r' be the radii of the given circles. Then since $hn=hm$, $hc-hc'=r-r'=$a constant.

From the definition of the curve the locus will be an hyperbola. Moreover this curve will be tangent to the lines connecting c and c' with the focoids or circular points at infinity (*Salmon's Conic Sections*, Art. 258; *Scott's Modern Analytical Geometry*, Art. 129).

Since this property is a descriptive one (*Scott's Modern Analytical Geometry*, Chapter V), it will be true after projection and the only difference between the proposition given at the beginning of this article and the one just proved is in the location of the points A and B. In the latter the circular points at infinity are the given points, while in the former any two points may be taken. The former is a general proposition, the latter a special case. In this special case we say "circle" instead of "conic through two fixed points." We say "center" for "pole of the line AB with respect to the conic," and again "hyperbola" for "conic touching the four lines CA, CB, $C'A$, and $C'B$.

Referring again to figure 1, we see that there may be two cases, one in which the two given conics intersect in real points and the other in which they

have an imaginery intersection. The two cases after projection are illustrated in figures 2, 3, 4 and 5. In figure 3, one conic lay within the other so that after projection one circle will be within the other and our locus is easily proved to be an ellipse with C and C' the centers of the given circles as foci. Moreover the ellipse is tangent to the lines connecting C and C' with the focoids (*Scott's Modern Analytical Geometry*, Arts. 129-130; *Salmon's Conic Sections*, Art. 258) Figures 4 and 5 show the case in which the circles after projection, intersect in real points. One set of circles will give an hyperbola and the other an ellipse as the locus. If in figures 2 and 3, the two given circles have equal radii, the locus of the light lined circles will be a straight line perpendicular to the line connecting the centers.

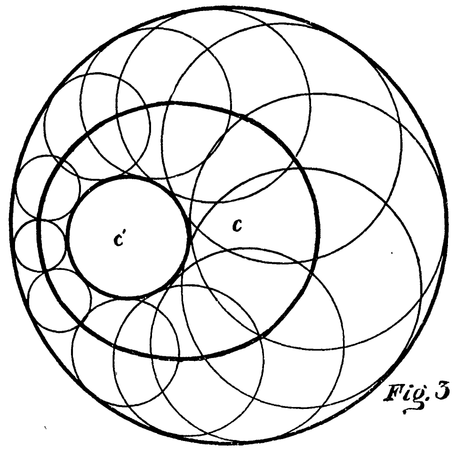

Fig. 3

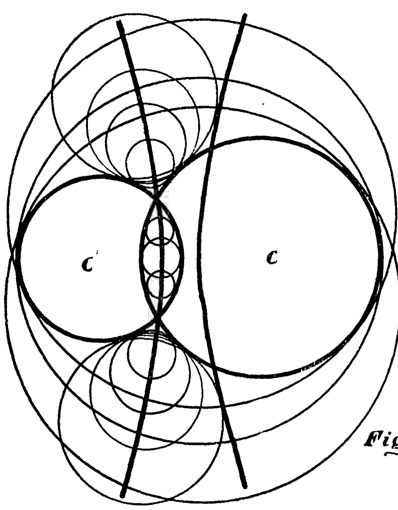

Fig. 4

If now we imagine one of the conics in figure 1 contracted to a point we shall have another proposition, viz:

If a conic be described tangent to a given conic and passing through two fixed points on that conic, and also passing through some other given point, the locus of the pole with respect to the varying conic of the chord connecting the first two points is a conic tangent to the lines connecting them with the third point, and also tangent to the lines connecting them with the pole of the chord with respect to the given conic.

This is another seemingly complicated and difficult proposition, but it is easily proved by projection. In figure 6, let A and B be the two points on the conic $ACDB$. Let P be the third given point. An infinite number of conics may be drawn through A, B and P and tangent to $ACDB$. The heavy lined conic in figure 6 is one of these. We must prove then that the locus of M as the variable conic changes is a conic tangent to PA, PB, NA and NB. As before, project A and B into the focoids and $ACDB$ in-

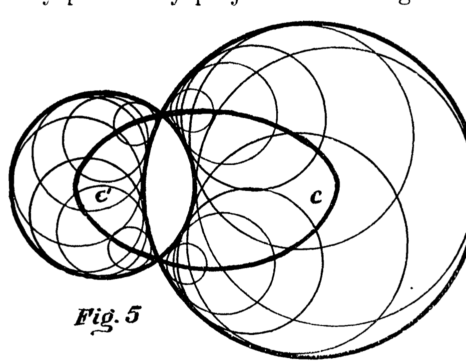

Fig. 5

to a circle. The system of conics through P, A, and B and tangent to $ACDB$ will then be a system of circles.

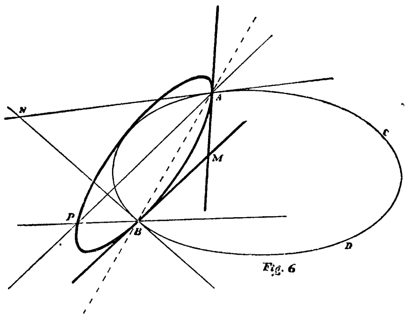

Fig. 6

There will be two cases in this proposition. We may take the given point within or without the conic. Figures 7 and 8 show the two cases after projection. P and C are the projections of P and N respectively. The proposition will now read: "The locus of the centers of a system of circles passing through a given point and tangent to a given circle is a conic having the given point and the center of the given circle for foci." In figure 7 when P is within the circle the locus is an ellipse. In figure 8, the locus is an hyperbola. We may have a third case if we imagine P upon the circumference of the circle. In that case the locus is a straight line through C and P.

Having proved this descriptive property after projection, it must be true before. Hence our general proposition is proved, and in the same manner that figures 2, 3, 4, and 5 are special cases of figure 1, so are figures 7 and 8 special cases of figure 6.

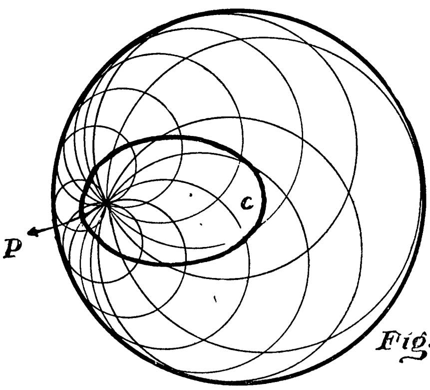

Fig. 7

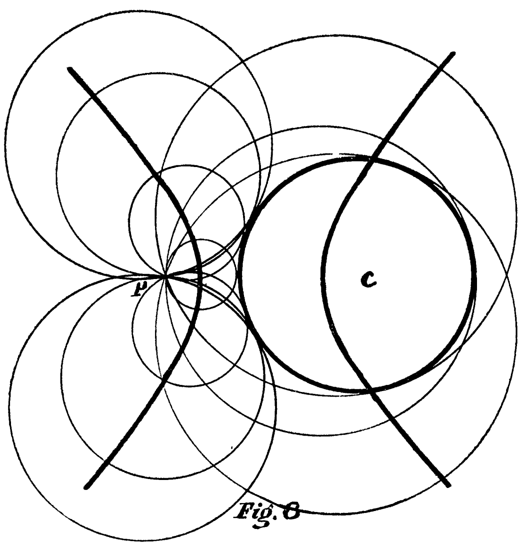

Fig. 8

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from November Number.]

CHAPTER IV.

COMBINATORY MULTIPLICATION.

35. Definition.—A product containing only units of the same system as factors and such that if the last two factors (called simple factors) are interchanged the sign of the product is changed is called *a combinatory product.*

Thus if E (not equal to 0) is a product of units and e_1, e_2 are units, and

$$[Ee_1e_2]+[Ee_2e_1]=0,$$

the product $[Ee_1e_2]$ is a combinatory product.

36. *In the combinatory product $[Abc]$ in which A is any product of a series of factors and b and c are simple factors if b and c are interchanged, the sign of the product is changed.*

Proof. 1. Suppose b and c at first to be *units.* Since A is any series of factors and these factors are numerically derivable from the units, we may write, after removing the coefficients (by 28), $A=\Sigma\alpha_rE_r$, where E_r are products of the units. Substituting

$$[Abc]+[Acb]\equiv[\Sigma\alpha_rE_r.bc]+[\Sigma\alpha_rE_r.cb]=\Sigma\alpha_r[E_rbc]+\Sigma\alpha_r[E_rcb] \ (29)$$

$$=\Sigma\alpha_r\{[E_rbc]+[E_rcb]\} \ (16,\ 11) =0 \ldots\ldots\ldots\ldots\ldots\ldots(35).$$

2. Next supposing b and c to be not units but numerically derivable from them. Let $b=\Sigma\beta_re_r$, $c=\Sigma\gamma_re_r$. Then

$$[Abc]+[Acb]\equiv[A.\Sigma\beta_re_r.\Sigma\gamma_re_r]+[A.\Sigma\gamma_re_r.\Sigma\beta_re_r]$$

$$=\Sigma\beta_r\gamma_s[Ae_re_s]+\Sigma\gamma_s\beta_r[Ae_se_r]\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\beta_r\gamma_s\{[Ae_re_s]+[Ae_se_r]\} \ (16) =0 \text{ (by 1 above).}$$

37. *In a combinatory product one can interchange any two successive simple factors providing the sign of the product be changed, that is to say*

$$[AbcD]+[AcbD]=0,$$

where A and D are any factor series, and b and c are simple factors.

$$[AbcD]+[AcbD]=[\{Abc\}D]+[\{Acb\}D] \text{ (13, Rem.)}$$

$$=[([Abc]+[Acb])D] \text{ (25) } =0 \ldots\ldots\ldots\ldots \text{ (36).}$$

38. *In a combinatory product one can interchange any two simple factors by changing the sign of the product.*

Thus, $P_{a,\,b}=-P_{b,\,a}$.

Proof.—Suppose n factors lie between a and b. Then n interchanges of adjacent factors will bring b into position next to a. After that $n+1$ interchanges of a with adjacent factors will but b in a's place and a in b's place. Thus there would be $2n+1$, or an odd number of changes of sign (37). Hence, ...

39. Definition.—If each of two series of quantities contain a and b once and but once and a stands *before* b in both or *after* b in both, then these quantities in those series are said to be *similarly arranged;* otherwise they are said to be *oppositely arranged.*

40. *Two combinatory products, which contain the same simple factors but in different order, are equal to each other or opposite in value according as the number of oppositely arranged pairs of factors is even or odd.*

Thus, $Q=(-1)^rP$, where P and Q are the two products and r is the number of oppositely arranged pairs of factors.

Proof.—If every pair of adjacent factors in Q were similarly arranged in P and Q, then, evidently, P and Q would be identical, and there would be no oppositely arranged pairs of factors in the two. If then there are oppositely arranged pairs of factors in P and Q, there must be at least one pair of factors adjacent in Q, which, as compared with the same in P, is oppositely arranged. Suppose after this pair of factors is interchanged in Q we call the result Q_1. Then $Q_1=-Q$. (37). Evidently P and Q_1 will have one less pair of oppositely arranged factor pairs than P and Q. Thus if r was the number at first, Q_1 and P will have $r-1$ such pairs. If r is not 1, there must be another such factor pair in Q_1 and P. Repeating the operation we get $Q_2=(-1)^2Q$. If therefore there were r oppositely arranged factor pairs at first,

$$Q=(-1)^rP.$$

41. *If B is a combinatory product containing r factors and C one containing s factors, then*

$$[ABC]=(-1)^{rs}[ACB].$$

Proof.—Let $C=c_1c_2 \ldots\ldots c_s$. Then since there will be a change of sign (37) each time c_1 interchanges with one of the r factors of B,

$$[ABc_1c_2\ldots\ldots c_s]=(-1)^r[Ac_1Bc_2c_3\ldots\ldots c_s]=(-1)^r(-1)^r[Ac_1c_2Bc_3\ldots\ldots c_s]$$

$$=(-1)^{sr}[Ac_1c_2c_3\ldots\ldots c_sB]=(-1)^{sr}[ACB].$$

42. *If A, B, C are products containing respectively q, r, s factors, then from the preceding article it is plain that*

$$[ABC]=(-1)^{rs+sq+qr}[CBA].$$

43. *If two simple factors of a combinatory product are equal, the product is zero.*

PROOF.—$P_{a,\,b}+P_{b,\,a}=0$ ....(38).

Then, if $b=a$, $P_{a,\,a}=0$.

44. *A combinatory product equals zero if a numerical relation exists between its simple factors.*

Let $a_1=\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots\alpha_m a_m$. Then

$$[a_1 a_2\ldots\ldots a_m]\equiv[(\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots\alpha_m a_m)a_2 a_3\ldots\ldots a_m]$$

$$=\alpha_2[a_2 a_2 a_3\ \ldots\ldots a_m]+\alpha_3[a_3 a_2 a_3\ldots\ldots a_m]+\ldots\ldots(29)$$

$$=0+0+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43).$$

45. *The combinatory product of n simple factors which are numerically derivable from the n quantities a_1, a_2, ... a_n equals the determinant formed from the numerical coefficients in their values times* $[a_1 a_2\ldots\ldots a_n]$. Thus

$$[(\alpha_{11}a_1+\ldots\ldots\alpha_{n1}a_n)(\alpha_{12}a_1+\ldots\ldots\alpha_{n2}a_n)\ldots\ldots(\alpha_{1n}a_1+\ldots\ldots\alpha_{nn}a_n)]$$

$$=\begin{vmatrix}\alpha_{11} & \alpha_{12}\ldots\ldots\alpha_{1n}\\ \alpha_{21} & \alpha_{22}\ldots\ldots\alpha_{2n}\\ \ldots\ldots & \ldots\ldots\ldots\ldots\\ \ldots\ldots & \ldots\ldots\ldots\ldots\\ \alpha_{n1}\ldots & \ldots\ldots\ldots\alpha_{nn}\end{vmatrix}[a_1 a_2 a_3\ldots\ldots a_n].$$

For, the product on the left side equals by (28),

$$\Sigma\alpha_{s1}\alpha_{t2}\ldots\ldots\alpha_{wn}[a_s a_t\ldots\ldots a_w],$$

where the n subscripts s, t, ...w assume in turn all the values from 1 to n. Now those terms of the distributed product in which two or more of the a's are equal disappear by (43). There remains then only those terms which contain all the a's used each once. We have now

$$[a_s a_t\ \ldots\ldots a_w]=(-1)^r[a_1 a_2 a_3\ldots\ldots a_n]\ldots\ldots\ldots\ldots(40),$$

where r denotes the number of pairs of factors in the left member oppositely arranged as compared with $[a_1 a_2 a_3\ldots\ldots a_n]$. But this is the law which determines the sign of any term of a determinant. See *e. g. Salmon's **Higher Algebra***, §8.

46. Definition.—By the *multiplicative combinations* of a series of quantities are meant those products which are their combinations without repetition. The simple factors are called the elements of the combination.

47. *Every combinatory product of m factors which are numerically expressed in terms of the n independent quantities a_1, a_2, ... a_n is numerically expressible in terms of the multiplicative combinations of the mth class of $a_1 \ldots\ldots a_n$, and each of these combinations has for its coefficient the determinant formed out of the m^2 numerical coefficients belonging to its m elements. Thus,*

$$[\Sigma\alpha_a a_a . \Sigma\beta_b b_b \ldots\ldots] = \Sigma \begin{vmatrix} \alpha_r & \cdots & \cdots \\ \vdots & \beta_s & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \ddots \end{vmatrix} [a_r a_s \ldots\ldots]$$

where $r < s < \ldots\ldots$

Proof.—Evident from that of Art. 45.

48. *If $a_1 \ldots$ a_n are independent, then their multiplicative combinations of any particular class are independent.*

Proof.—Let $\alpha A + \beta B + \ldots\ldots = 0$, in which A, B, ... are the multiplicative combinations of any one class formed out of $a_1 \ldots\ldots a_n$, and α, β...... are numbers. Let ns multiply the equation through by A', the product of all the factors not found in A. Then B, C, ... would each contain one or more of the elements of A', and the products $[BA']$, $[CA']$, ... would each equal zero (43). Then we have $\alpha[AA'] = 0$. Now, $[AA']$ is not equal to zero. Hence $\alpha = 0$. In the same way we can prove that β, γ, ... must each equal zero. Hence there can be no such equation as $\alpha A + \beta B + \ldots\ldots = 0$, which expresses a dependence between A, B, ... Thus, A, B, ... are independent.

49. *A combinatory product remains constant when to any simple factor an arbitrary multiple of another is added.*

Proof.—$P_{a,\, b+qa} = P_{a,\, b} + qP_{a,\, a}$ (29) $= P_{a,\, b}$ (43).

50. Definition.—If from a series of quantities a second is derived by adding to any quantity a multiple of an adjacent quantity, then the first series is said to be changed into the second by a *simple linear alteration.* If the operation is repeated it is called a *multiple linear alteration.*

From what we saw in 49, it appears that the value of a quantity is not affected by linear alteration.

51. Definition.—The multiplicative combinations of the original units of the mth class is called a *unit* of the mth order, and a quantity numerically derived from such units is called a *quantity* of the mth order.

The space derived from the simple factors of a quantity (17) is called the space of this quantity. A quantity is subordinate to another if its space is.

52. Definition.—The outer product of two *units* of a higher order is obtained by merely uniting their simple factors into a combinatory product.

Thus, $[(e_1e_2\ldots\ldots e_m)(e_{m+1}\ldots\ldots e_n)]=[e_1e_2\ldots\ldots e_n]$.

53. *In order to multiply two simple quantities*, $[ab\ldots\ldots]$ and $[cd\ldots\ldots]$, *it is sufficient to unite their simple factors taken in order into a single combinatory product* $[ab\ldots\ldots cd\ldots\ldots]$.

PROOF.—Let $e_1\ldots\ldots e_n$ be the original units, and let $a=\Sigma\alpha_ae_a$, $b=\Sigma\beta_be_b$, $c=\Sigma\gamma_ce_c$, $d=\Sigma\delta_de_d$. Then

$$[(ab\ldots\ldots)(cd\ldots\ldots)]\equiv[(\Sigma\alpha_ae_a.\Sigma\beta_be_b\ \ldots\ldots)(\Sigma\gamma_ce_c.\Sigma\delta_de_d\ldots\ldots)]$$

$$=[\Sigma\{\alpha_a\beta_b\ldots\ldots[e_ae_b]\}\Sigma\{\gamma_c\delta_d\ldots\ldots[e_ce_d]\}]\ \ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\{\alpha_a\beta_b\ldots\ldots\gamma_c\delta_d\ldots\ldots[(e_ae_b\ldots\ldots)(e_ce_d\ldots\ldots)]\}\ \ldots\ldots(28)$$

$$=\Sigma\{\alpha_a\beta_b\ldots\ldots\gamma_c\delta_d\ldots\ldots[e_ae_b\ldots\ldots e_ce_d\ldots\ldots]\}\ \ldots\ldots(52)$$

$$=[\Sigma\alpha_ae_a.\Sigma\beta_be_b\ldots\ldots\Sigma\gamma_ce_c.\Sigma\delta_de_d\ldots\ldots]\ \ldots\ldots\ldots\ldots(28)$$

$$\equiv[ab\ldots\ldots cd\ldots\ldots].$$

54. COROLLARY TO 53.—If a simple quantity A is subordinate to B (18), then B may be written $B=[AC]$, where C is a simple factor.

55. *To show that* $[A(BC)]=[ABC]$, *i. e. to show that the associative law holds.*

PROOF.—1. When A, B, and C are the products of simple factors, the truth of this case follows readily from 53.

2. When A, B, and C are sums of simple quantities, $A=\Sigma A_a$, $B=\Sigma B_b$, $C=\Sigma C_c$.

$$[(A(BC)]\equiv[\Sigma A_a.(\Sigma B_b\Sigma C_c)]=\Sigma[A_a(B_bC_c)]\ \ldots\ldots\ldots\ldots(28)$$

$$=\Sigma[A_aB_bC_c]\text{ (By 1, above) }=[\Sigma A_a.\Sigma B_b.\Sigma C_c]\text{ (28) }=[ABC].$$

[To be Continued.]

A SOLUTION OF THE OBLIQUE TRIANGLE GIVEN TWO SIDES AND THE INCLUDED ANGLE.

By H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Take the data to be b, c, A.

$$p(\cot A+\cot B)=c.$$

Divide by $p=b\sin A$ and transpose $\cot A$.

$$\cot B=\frac{c}{b\sin A}-\cot A\ ;\ a=\frac{b\sin A}{\sin B}.$$

Find logarithm of $b\sin A$ in the column for a; then in the column for B, find $\frac{c}{b\sin A}$ as a logarithm, then as a number; to find B, use the table of natural cotangents.

If a solution wholly logarithmic is desired, take

$$\tan\theta=\frac{b\sin A}{c}\ ;\ \tan B=\frac{\sin A\sin\theta}{\sin(A-\theta)};\ a=\frac{b\sin A}{\sin B}.$$

INTEGRATION OF ELLIPTIC INTEGRALS.

By GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High Schoool, Chester, Pa.

[Continued from November Number.]

Since all the expressions in the form of

$$\int_0^{2\pi}\sin^a\theta\cos^{2b+1}\theta(1-e^2\sin^2\theta)^c d\theta=0,$$

(where a, b, c are integers), we will not consider them in our subsequent treatment. From (13_0) we have,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{e^2}[F(e,\ \tfrac{1}{2}\pi)-E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(3)$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{e^2}[E(e,\ \tfrac{1}{2}\pi)-(1-e^2)F(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ldots\ldots(4).$$

If $n=0$ in (10_0), $A_2=\frac{2}{3}[(1+e^2)/e]A_1-\frac{1}{3}A_0$.

$$\therefore A_2=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos2\varphi d\varphi}{(1+e^2-2e\cos\theta)^{\frac{1}{2}}}$$

$$=\frac{4}{3\pi e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(14_0).$$

$$\therefore \int_0^{2\pi}\frac{4e^2\sin^4\theta+1-2(1+e^2)\sin^2\theta}{\sqrt{(1-e^2\sin^2\theta)}}d\theta$$

$$=\frac{4}{3e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(15_0).$$

$$\therefore\ 4e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-2(1+e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)].$$

$\therefore$ From (1) and (3) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{3e^4}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots\ (5).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$\therefore$ From (3) and (5) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{3e^4}[(2-e^2)E(e,\ \tfrac{1}{2}\pi)-2(1-e^2)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(6).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$\therefore$ From (4) and (6) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{3e^4}[(2-5e^2+3e^4)F(e,\ \tfrac{1}{2}\pi)-2(1-2e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(7).$$

(15_0) may be written as follows :

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-2\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta-2e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)].$$

$\therefore$ From (1) and (6) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta=\frac{1}{3e^2}[(1-e^2)F(e,\ \tfrac{1}{2}\pi)-(1-2e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots(8).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta=\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}d\theta-\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^2\theta d\theta.$$

$\therefore$ From (2) and (8) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^2\theta d\theta=\frac{1}{3e^2}[(1+e^2)E(e,\ \tfrac{1}{2}\pi)-(1-e^2)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(9).$$

Let $n=1$ in (10_0), then $A_3=\frac{4}{5}.\frac{1+e^2}{e}.A_2-\frac{3}{5}A_1$.

$$\therefore\ A_3=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos 3\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}}$$

$$=\frac{4}{15\pi e^3}[(8+3e^2+4e^4)F(e,\ \tfrac{1}{2}\pi)-(8+7e^2+8e^4)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(16_0).$$

From (4_0) we get

$$16e^3\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+12e\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-12e^3\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-3e\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^3}[(8+3e^2+4e^4)F(e,\ \tfrac{1}{2}\pi)$$

$$-(8+7e^2+8e^4)E(e,\ \tfrac{1}{2}\pi)\ldots\ldots(17_0).$$

$\therefore$ From (3), (6), (7) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^6}[(8+3e^2+4e^4)F(e,\ \tfrac{1}{4}\pi)$$

$$-(8+7e^2+8e^4)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(10).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$\therefore$ From (5) and (10) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^6}[(8-3e^2-2e^4)E(e,\tfrac{1}{2}\pi)-(8-7e^2-e^4)F(e,\tfrac{1}{2}\pi)].(11)$$

$$\int_0^{2\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{2\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

∴ From (6) and (11) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^6}[(8-17e^2+9e^4)F(e,\tfrac{1}{2}\pi)$$

$$-(8-13e^2+3e^4)E(e,\tfrac{1}{2}\pi)]\ \ldots\ldots(12).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

∴ From (7) and (12) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^6}[(8-23e^2+23e^4)E(e,\tfrac{1}{2}\pi)$$

$$-(8-27e^2+34e^4-15e^6)F(e,\tfrac{1}{2}\pi)]\ldots\ldots(13).$$

(17_0) may be written as follows :

$$4e^3\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+12e\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^2\theta d\theta$$

$$-3e\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^3}[(8+3e^2+4e^4)F(e,\tfrac{1}{2}\pi)-(8+7e^2+8e^4)E(e,\tfrac{1}{2}\pi)].$$

∴ From (10) and (3) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^2\theta d\theta$$

$$=\frac{1}{15e^4}[2(1-e^2+e^4)E(e,\tfrac{1}{2}\pi)-(2-3e^2+e^4)F(e,\tfrac{1}{2}\pi)]\ldots\ldots(14).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^2\theta d\theta=\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta$$

$$-\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^4\theta d\theta=\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^2\theta d\theta$$

$$-\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^4\theta d\theta.$$

From (8), (14) and (9), (14) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^4\theta d\theta = \frac{1}{15e^4}[2(1+e^2-2e^4)F(e,\ \tfrac{1}{2}\pi)$$

$$-(2+3e^2-8e^4)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (15).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^4\theta d\theta = \frac{1}{15e^4}[2(1-4e^2+3e^4)F(e,\ \tfrac{1}{2}\pi)$$

$$-(2-7e^2-3e^4)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (16).$$

Let $n=2$ in (10_0), then $A_4 = \frac{6}{7}.\frac{1+e^2}{e}.A_3 - \frac{5}{7}A_2$.

$$\therefore\ A_4 = \frac{4}{105\pi e^4}[(48+16e^2+17e^4+24e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$-8(6+5e^2+5e^4+6e^6)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (18_0).$$

$\therefore$ By symmetry from A_2, A_3 we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{105e^8}[(48+16e^2+17e^4+24e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$-8(6+5e^2+5e^4+6e^6)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (17).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^8\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$\therefore$ From (10) and (17) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{105e^8}[(48-16e^2-9e^4-8e^6)E(e,\ \tfrac{1}{2}\pi)$$

$$-4(12-10e^2-e^4-e^6)F(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (18).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$\therefore$ From (11) and (18) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{35e^8}[(16-32e^2+15e^4+e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$-2(8-12e^2+2e^4+e^6)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (19).$$

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

I saw problem 117 in a copy of *Ostrander's Arithmetic* in the winter of 1849-50, when I was nearly eleven years old, and solved it, at that time, for my teacher, Mr. John A. Shearer. Said Arithmetic was probably first published many years before that time. I herewith reproduce with the data of Problem 117 my *arithmetic* solution :

Consider an arc ab=20 inches=$\frac{10}{99}$ of a rod, c being the center of the circular farm. Then as sector acb=an acre=160 rods, we have, $r=160\div\frac{5}{99}=32\times 99=320\times 9.9=9.9$ miles.

Since the ratio of the areas of a circle and circumscribed square is equal to the ratio of their perimeters, the diameter of the required square is equal to that of the circle, or $9.9\times 2=19.8$ miles.

The areas of the tracts are easily found, and agree with the results already given ; except, that in the first solution, 3068 is given in place of 3168, which leads to an area too small.

If we lay off ab=20 inches=$\frac{10}{99}$ of a rod on one side of the square (side=$2x$), c being the center, we have as for the circle, $x=160\div\frac{5}{99}=9.9$ miles, or $2x$= 19.8 miles. But I found the side of the square, as first shown above, in my long-ago solution.

It may be worth remarking that the version of Mr. Baker differs slightly from that of Ostrander, as I remember it. P. H. PHILBRICK.

THE BEGINNING OF THE 20TH CENTURY.

The question as to when the 20th century begins has received so much attention in the last three or four years, and especially within the last six months —articles having appeared in a number of the leading magazines—that a brief notice of it in the MONTHLY may not be out of place.

It is strange, indeed, that so simple a question as to the time when the 20th century begins should occasion any controversy ; and yet there has been a wide and somewhat general misunderstanding about it. A few years ago, a circircular was sent out by one of our leading book publishing companies, in which circular it was stated that the 20th century would begin December 31, 1901. Of course, this may have been a typographical error. But the general belief is that the 20th century will begin at 12 A. M., January 1, 1900. Such, however, is not the case, since up to that time, only 99 years of the present century will have passed. The 19th century will end at 12 o'clock P. M., December 31st, 1900, and the 20th century will begin at 12 o'clock A. M., January 1, 1901.

The misunderstanding of the matter has probably arisen from neglecting

to note that in the process of counting, we do not begin with 0. There never was a year 0. In dating our letters, we make clear this point. Suppose that on the second day after the Christian Era, a letter had been dated; using our present conventions, the letter would have been dated January 2, 1; and this would mean 2 days of year one. So December 30, 1, would mean 364 days of year 1; and not until midnight of December 31, was the year 1 completed. As with the year, so with the century. December 30, 99, means 364 days, or $\frac{364}{365}$ of the 99th year, or 98 years$+\frac{364}{365}$ of 1 year; December 27, 1899, means 361 days of the 1899th year since the Christian Era, or 1898 years+361 days or 1898 years$+\frac{361}{365}$ of 1 year, or the 361st day of the 99th year of the 19th century.

B. F. FINKEL.

ALGEBRA.

95. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Substitute *numbers* in place of the letters in the following pattern: $\triangle=\sqrt{(81^2a^2b^2c^2)}=81abc\ldots\ldots b^2+c^2,\ a^2+c^2,\ a^2+b^2$; and compute the areas and sides of the whole nest of integral, rational triangles.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Sides of general triangle are a^2+b^2, a^2+c^2, and b^2+c^2.

Whence $\triangle=\sqrt{[(a^2+b^2+c^2)a^2b^2c^2]}$.

$\therefore\ 81^2=a^2+b^2+c^2=$the sum of three squares.

When $(3^n)^2=$the sum of three integral squares, I have found that $\dfrac{3^n+2n-1}{4}=$the *number of sets* of three squares, of which $\dfrac{3^{n-1}+1}{2}$ are prime sets and $\dfrac{3^{n-1}+2n-3}{4}$ are multiple sets.

But $81^2=(3^4)^2$; whence $\dfrac{3^n+2n-1}{4}=22$.

$\therefore$ There are 22 sets of three integral squares whose respective sums$=81^2$.

$\therefore$ Also, there are 22 different integral, rational triangles whose respective areas$=81abc$.

Following are the 22 sets of three integers the sum of whose squares is 81^2, $=6561$:

1, 28, 76; 1, 44, 68; 6, 21, 78; 6, 30, 75; 6, 42, 69; 8, 16, 79; 8, 49, 64; 9, 36, 72; 16, 23, 76; 16, 41, 68; 16, 47, 64; 17, 56, 56; 20, 44, 65; 20, 55, 56; 21, 42, 66; 23, 44, 64; 27, 54, 54; 28, 41, 64; 30, 30, 69; 32, 49, 56; 36, 36, 63; and 40, 44, 55.

From these we derive the following 22 integral, rational triangles:

(1). Sides=785 $[=1^2+28^2]$, 5777 $[=1^2+76^2]$, 6560 $[=28^2+76^2]$; area $=172368=81\times1\times28\times76$.

(2). Sides=1937, 4625, 6560; area=242352.

(3). Sides=477, 6120, 6525; area=796068.

(4). Sides=936, 5661, 6525 ; area=1093500.
(5). Sides=1800, 4797, 6525 ; area=1408428.
(6). Sides=320, 6305, 6497 ; area=819072.
(7). Sides=2465, 4160, 6497 ; area=2032128.
(8). Sides=1377, 5265, 6480 ; area=1889568.
(9). Sides=785, 6032, 6305 ; area=2265408.
(10). Sides=1937, 4880, 6305 ; area=3613248.
(11). Sides=2465, 4352, 6305 ; area=3898368.
(12). Sides=3425, 3425, 6272 ; area=4318272.
(13). Sides=2336, 4625, 6161 ; area=4633200.
(14). Sides=3425, 3536, 6161 ; area=4989600.
(15). Sides=2205, 4797, 6120 ; area=4715172.
(16). Sides=2465, 4625, 6032 ; area=5246208.
(17). Sides=3645, 3645, 5832 ; area=6377292.
(18). Sides=2465, 4880, 5777 ; area=5951232.
(19). Sides=1800, 5661, 5661 ; area=5030100.
(20). Sides=3425, 4160, 5537 ; area=7112448.
(21). Sides=2592, 5265, 5265 ; area=6613488.
(22). Sides=3536, 4625, 4961 ; area=7840800.

Also solved by *ELMER SCHUYLER*, and *CHAS. C. CROSS*.

96. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

How many different numbers may be written with the nine digits and zero, using them singly and in groups of from one to ten digits each, and using no figure but once in each group? How many more numbers may be written by repeating the digits and zero at pleasure in each group?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and C. B. GOULD, Colorado College, Colorado Springs, Col.

We will not regard zero as a number.

(1). ${}^nP_r$ will be used to denote the number of permutations of n things taken r at a time.

Then ${}^{10}P_1=10$. This includes zero.

$\therefore$ ${}^{10}P_1-1=9$ when taken singly.

${}^{10}P_2=90$. This includes those that have zero for the first digit, and were therefore included in the first group ,*i. e.* 9 is the sama as 09.

Hence we proceed as follows :

${}^{10}P_1-1=10-1=9$, taken 1 at a time.

${}^{10}P_2-{}^9P_1=90-9=81$, taken 2 at a time.

${}^{10}P_3-{}^9P_2=720-72=648$, taken 3 at a time.

${}^{10}P_4-{}^9P_3=5040-504=4536$, taken 4 at a time.

${}^{10}P_5-{}^9P_4=30240-3024=27216$, taken 5 at a time.

${}^{10}P_6-{}^9P_5=151200-15120=136080$, taken 6 at a time.

${}^{10}P_7-{}^9P_6=604800-60480=544320$, taken 7 at a time.

${}^{10}P_8-{}^9P_7=1814400-181440=1632960$, taken 8 at a time.

${}^{10}P_9 - {}^9P_8 = 3628800 - 362880 = 3265920$, taken 9 at a time.

${}^{10}P_{10} - {}^9P_9 = 3628800 - 362880 = 3265920$, taken 10 at a time.

$\therefore$ 8877690 in all.

(2). This is equivalent to asking how many numbers less than 10000000000 can be made with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

$\therefore (10)^{10} - 1 = 9999999999$.

$9999999999 - 8877690 = 9991122309$ more by the second arrangement.

Mr. Gould's solution is similar to the above, but by including Mr. Zerr's he gets as results 8,877,691 and 10,000,000,000.

EDITORIALS.

The Index to Vol. VI was prepared by Prof. J. Scheffer.

The MONTHLY and any of the leading magazines may be procured at reduced rates.

This number concludes Vol. VI of the MONTHLY. We take this opportunity to thank our contributors for their generous coöperation in making the journal a success; and we hope we may rely on their help during the year 1900. In order that the subscription list may be increased we make this offer: An old subscriber sending us $3., may have his own subscription and that of a new subscriber credited up to January 1, 1901.

The Report on Progress in Non-Euclidean Geometry which appeared in THE AMERICAN MATHEMATICAL MONTHLY for October, and which is contained in full in the Proceedings of the American Association for the Advancement of Science, has also been published in full in *Science* for October 20, and now appears in *Popular Astronomy* for November and December. An editorial, addressed to "Teachers of Geometry and Astronomy," says: "We have printed two articles in this number to which special attention is asked on the part of teachers of Geometry and Astronomy. . . .

The other article is by Professor George Bruce Halsted, who, not long ago, was asked by the American Association for the Advancement of Science to prepare a report which was given at its Columbus meeting showing recent progress in non-Euclidean Geometry.

Professor Halsted has done the teachers of Geometry most excellent service in the preparation of this paper. We call attention to this early, that teachers may read it thoroughly, so as to be acquainted with what is now going on among the masters in pure mathematics. The information gained will be helpful to those who want to be abreast with the best teaching talent of the present time."

PROBLEMS FOR SOLUTION.

ALGEBRA.

113. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find a square number consisting of 24 figures such that the numbers formed by the first twelve figures and the last twelve figures, respectively, are consecutive, and *vice versa.*

114. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

$$\left.\begin{array}{l} bx^3=10a^2bx+3a^3y \\ ay^3=10b^2y+3b^3x \end{array}\right\}$$

*** Solutions of these problems should be sent to J. M. Colaw not later than February 10.

CALCULUS.

104. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the differential equation corresponding to $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=[a(x-y)]$.

105. Proposed by CHAS. C. CROSS, Whaleyville, Va.

From all points in a straight line passing through the center of a given circle tangents are drawn to the circle. If the bases and vertices of all the angles thus formed are made to coincide; required the equation of the curve passing through the langent points.

*** Solutions of these problems should be sent to J. M. Colaw not later than February 10.

MISCELLANEOUS.

83. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Segment's area of a circle whose radius is 5 inches is 28.56 square inches. Find the chord.

84. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Provė that $e^\pi+e^{-\pi}=2[1+2^2][1+(\frac{2}{3})^2][1+(\frac{2}{5})^2]\ldots\ldots$ ad infinitum.

*** Solutions of these problems should be sent to J. M. Colow not later than February 10.